Geometry and topology optimization of plane frame using force density method

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Compliance minimization Plane frame

1 Introduction

Geometry and topology optimization of truss-like structures is one of the standard topics in structural optimization, in which sizing optimization is usually included[1]. Since nodal locations are fixed in the ground structure method, it is reasonable to incorporate nodal coordinates as variables to simultaneously optimize geometry and topology of truss-like structures. However, it is difficult to modify structural topology in the process of geometry optimization because of the existence of coalescent nodes, causing singularity in stiffness matrix[2]. Therefore, appropriately applying side constraints on nodal locations for preventing melting nodes is a challenging problem.

This study presents a method to resolve such problem by using force density method (FDM). The nodal coordinates of frames are computed as functions of force densities of an auxiliary cable net that has support conditions different from the frame to be optimized.

2 Frame model

Consider a plane frame discretized by beam elements with n nodes and m members. The stiffness equation is written as

$$\boldsymbol{KU}=\boldsymbol{F} \tag{1}$$

where $\boldsymbol{K}$ is the structural stiffness matrix, and $\boldsymbol{U}$ and $\boldsymbol{F}$ are the vectors of nodal displacements and external loads, respectively. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ denote the vectors of x- and y- coordinates of nodes, and assume each member has circular cross-section. Eq. (1) can then be written as

$$\boldsymbol{K}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{d})\boldsymbol{U}=\boldsymbol{F} \tag{2}$$

where $\boldsymbol{d}$ is the diameter vector of beam elements.

3 Force density method

Although a plane frame is to be optimized, its nodal locations are defined in terms of force densities of an auxiliary cable net. Let $\boldsymbol{C}$ denote the connectivity matrix and $\boldsymbol{q}$ denote the force density vector. The force density matrix $\boldsymbol{Q}$ is given as[3,4]

$$\boldsymbol{Q}=\boldsymbol{C}^T\mathrm{diag}(\boldsymbol{q})\boldsymbol{C} \tag{3}$$

where diag($\boldsymbol{q}$) represents the diagonal matrix of $\boldsymbol{q}$. Support conditions of the cable net are different from those of the frame to be optimized, and the loaded nodes of the frame are included in the fixed nodes of the cable net. Let $\boldsymbol{x}_{\mathrm{free}}$, $\boldsymbol{y}_{\mathrm{free}}$ and $\boldsymbol{x}_{\mathrm{fix}}$, $\boldsymbol{y}_{\mathrm{fix}}$ denote the x- and y-coordinate vectors of free nodes and fixed nodes, respectively. Arrange the columns of $\boldsymbol{C}$ such that the columns corresponding to the free nodes precede those corresponding to the fix nodes, i.e., $\boldsymbol{C}$ = ($\boldsymbol{C}_{\mathrm{free}}$, $\boldsymbol{C}_{\mathrm{fix}}$). The equilibrium equations of free nodes and fix nodes are written as

$$\begin{aligned}\boldsymbol{C}_{\mathrm{free}}^T\mathrm{diag}(\boldsymbol{q})\boldsymbol{C}_{\mathrm{free}}\boldsymbol{x}_{\mathrm{free}}+\boldsymbol{C}_{\mathrm{free}}^T\mathrm{diag}(\boldsymbol{q})\boldsymbol{C}_{\mathrm{fix}}\boldsymbol{x}_{\mathrm{fix}}=\boldsymbol{0}\\ \boldsymbol{C}_{\mathrm{free}}^T\mathrm{diag}(\boldsymbol{q})\boldsymbol{C}_{\mathrm{free}}\boldsymbol{y}_{\mathrm{free}}+\boldsymbol{C}_{\mathrm{free}}^T\mathrm{diag}(\boldsymbol{q})\boldsymbol{C}_{\mathrm{fix}}\boldsymbol{y}_{\mathrm{fix}}=\boldsymbol{0}\end{aligned} \tag{4}$$

Therefore, coordinates of free nodes can be considered as functions of $\boldsymbol{q}$.

4 Optimal problem

The optimization problem for minimizing the compliance under volume constraints is formulated as follows:

$$\begin{aligned}&\text{Minimize } F(\boldsymbol{x}_{\mathrm{free}},\boldsymbol{y}_{\mathrm{free}},\boldsymbol{d})=\boldsymbol{U}^T\boldsymbol{K}(\boldsymbol{x}_{\mathrm{free}},\boldsymbol{y}_{\mathrm{free}},\boldsymbol{d})\boldsymbol{U}\\ &\text{subject to } V(\boldsymbol{d},\boldsymbol{x}_{\mathrm{free}},\boldsymbol{y}_{\mathrm{free}})\le V_{\mathrm{upper}}\end{aligned} \tag{5}$$

with side constraints

$$\begin{aligned}&x_{j,\mathrm{lower}}\le x_j\le x_{j,\mathrm{upper}},\quad y_{j,\mathrm{lower}}\le y_j\le y_{j,\mathrm{upper}},\quad j=1,2,\cdots n_{\mathrm{free}}\\ &d_{i,\mathrm{lower}}\le d_i\le d_{i,\mathrm{upper}},\quad i=1,2,\cdots m\end{aligned} \tag{6}$$

where V_{upper} is the specified upper bound for volume, n_{free} is the number of free nodes, and the subscripts 'upper' and 'lower' indicate the upper and lower bounds, respectively. Since coordinates of free nodes can be obtained by solving Eq. (4), optimization problem (5) is restated as

$$\begin{aligned}&\text{Minimize } F(\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}),\boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q}),\boldsymbol{d})=\boldsymbol{U}^T\boldsymbol{K}(\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}),\boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q}),\boldsymbol{d})\boldsymbol{U}\\ &\text{subject to } V\left(\boldsymbol{d},\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}),\boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q})\right)\le V_{\mathrm{upper}}\end{aligned} \tag{7}$$

with following side constraints

$$\begin{aligned}&x_{j,\mathrm{lower}}\le x_j\le x_{j,\mathrm{upper}},\quad y_{j,\mathrm{lower}}\le y_j\le y_{j,\mathrm{upper}},\quad j=1,2,\cdots n_{\mathrm{free}}\\ &d_{i,\mathrm{lower}}\le d_i\le d_{i,\mathrm{upper}},\quad q_{i,\mathrm{lower}}\le q_i\le q_{i,\mathrm{upper}},\quad i=1,2,\cdots m\end{aligned} \tag{8}$$

It can be observed that optimal solutions of problems (5) and (7) are the same if a set of $\boldsymbol{q}$ in (5) can define the optimal solution of problem (7). According to the definition of force density, side constraints for q_i in Eq. (8) can be rewritten as

$$q_{i,\mathrm{lower}}\le\frac{N_i}{L_i}\le q_{i,\mathrm{upper}},\quad i=1,2,\cdots m \tag{9}$$

where N_i and L_i are the axial force and length of ith member, respectively. Assuming $q_{i,\mathrm{lower}}$ is a small positive value ε and $q_{i,\mathrm{upper}}$ is sufficiently large, Eq. (9) can be rewritten as

$$\frac{N_i}{q_{i,\mathrm{upper}}} \le L_i \le \frac{N_i}{\varepsilon}, \quad i=1,2,\cdots m \tag{9}$$

if N_i is not equal to 0. Hence, the length of ith element is indirectly controlled by side constraints of the corresponding force density. It is known that Eq. (4) has a solution if $q_i > 0$ for all elements.

5 Sensitivity analysis

The sensitivity coefficients of objective and constraint functions are obtained as

$$\frac{\partial F}{\partial x_{j,\mathrm{free}}} = -\boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial x_{j,\mathrm{free}}}\boldsymbol{U},\ \frac{\partial F}{\partial y_{j,\mathrm{free}}} = -\boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial y_{j,\mathrm{free}}}\boldsymbol{U},\ j=1,\cdots n_{\mathrm{free}} \tag{10}$$

$$\frac{\partial F}{\partial d_i} = -\boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial d_i}\boldsymbol{U}, \quad i=1,2,\cdots m \tag{11}$$

$$\frac{\partial V}{\partial q_i} = \sum_{i=1}^{m}\sum_{k=1}^{2} \frac{\partial L_i}{\partial x_{k,i}} \bullet \frac{\partial x_{k,i}}{\partial q_i} + \frac{\partial L_i}{\partial y_{k,i}} \bullet \frac{\partial y_{k,i}}{\partial q_i}, \quad \frac{\partial V}{\partial d_i} = \sum_{i=1}^{m} L_i \frac{\partial A_i}{\partial d_i} \tag{12}$$

where $(\cdot)_{k,i}$ (i=1,2) represents the two end nodes of ith element. Derivatives of coordinates of free nodes with respect to $\boldsymbol{q}$ are explicitly calculated by directly differentiating both sides of Eq. (4). Therefore, sensitivity coefficients of objective and constraint functions with respect to $\boldsymbol{q}$ and $\boldsymbol{d}$ are computed, and will be used in sequential quadratic programming (SQP) to solve problem (7).

6 Further optimization

The layout found by solving (7) may be unclear due to the existence of thin members and closely spaced nodes. Therefore, we further optimize $\boldsymbol{d}$, $\boldsymbol{x}_{\mathrm{free}}$ and $\boldsymbol{y}_{\mathrm{free}}$ by solving the following problem:

$$\begin{aligned} &\text{Minimize} \quad F(\boldsymbol{x}_{\mathrm{free},r}, \boldsymbol{y}_{\mathrm{free},r}, \boldsymbol{d}) = \boldsymbol{U}^T \boldsymbol{K}(\boldsymbol{x}_{\mathrm{free},r}, \boldsymbol{y}_{\mathrm{free},r}, \boldsymbol{d})\boldsymbol{U} \\ &\text{subject to } V\left(\boldsymbol{d}, \boldsymbol{x}_{\mathrm{free},r}, \boldsymbol{y}_{\mathrm{free},r}\right) \le V_{\mathrm{upper}} \end{aligned} \tag{13}$$

with side constraints

$$\begin{aligned} &d_{i,\mathrm{lower}} \le d_i \le d_{i,\mathrm{upper}}, \quad i=1,2,\cdots m_r \\ &x_{j,\mathrm{lower}} \le x_j \le x_{j,\mathrm{upper}}, \quad y_{j,\mathrm{lower}} \le y_j \le y_{j,\mathrm{upper}}, \quad j=1,2,\cdots n_{\mathrm{free},r} \end{aligned} \tag{14}$$

where $n_{\mathrm{free},\,r}$ and m_r are the numbers of free nodes and elements, respectively, after merging the closely spaced nodes of the optimal solution of problem (7). Same procedure of sensitivity analysis is also carried out as described in Section 5.

7 Example

A cantilever frame is optimized to verify the proposed method. The initial ground structure, shown in Fig. 1, is a 3×2 rectangular grid with 12 nodes and 27 elements, which is pin-supported at nodes 1, 2 and 3, and a single load P is applied at node 11. Accordingly, these four nodes are considered as fixed nodes. Note that elastic modulus E is the same for all members.

The upper-bound volume V_{upper} is equal to 1, ε is 0.0001, $q_{i,\mathrm{upper}}$ is 1000, $d_{i,\mathrm{lower}}$ is 0.001, and $d_{i,\mathrm{upper}}$ is not given. Fig. 1 shows the optimal solutions found by solving problems (7) and (13).

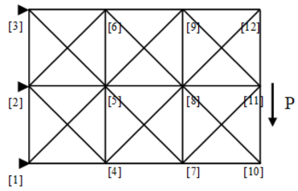

Fig. 1 Initial ground structure of cantilever frame.

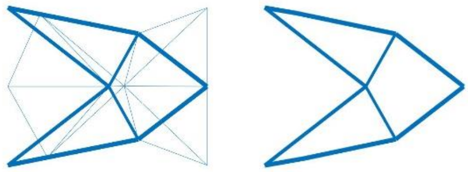

Fig. 2 Solutions before (left) and after (right) further optimization

As seen from the left part of Fig.2, no extremely short member exists, preventing singularity in stiffness matrix. Then, the optimal solution is further optimized by solving problem (13), with removal of thin elements, and the final results is shown in right part of Fig.2. Compliance of optimal solutions before and after further optimization are 83.2042 and 82.0946, respectively, showing good convergence of the proposed method.

Conclusion

A new optimization method for plane frame is presented using FDM. Nodal locations can be determined by a set of linear equations with respect to element force density, and the problem of determining the nodal location in geometry optimization can be therefore solved by finding the corresponding force density values of the auxiliary cable net. Side constraints of limiting element length is achieved by the side constraints of force density, preventing difficulties of coalescent nodes. An example of cantilever frame is discussed to illustrated the effectiveness of the proposed method.

Reference

[1] M. Ohsaki. Optimization of Finite Dimensional Structures, CRC Press. 2010.

[2] W. Achtziger, On simultaneous optimization of truss geometry and topology, Struct. Multidisc. Optim., Vol. 33, No. 4-5, pp. 285-304, 2007.

[3] J. Y. Zhang. and M. Ohsaki, Adaptive force density method for form-finding problem of tensegrity structure, Int. J. Solids Struct., Vol. 43, No. 18-19, pp. 5658-5673, 2006.

[4] M. Ohsaki and K. Hayashi, Force density method for simultaneous optimization of geometry and topology of trusses, Struct. Multidisc. Optim., Vol. 56, No. 5 pp. 1157-1168, 2017.

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