Sequential optimization and reliability assessment for shape and topology of plane frames using L-moments

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Quantile function Sample L-moments
Force density method Maximum entropy method

1 Introduction

Sequential optimization and reliability assessment (SORA) is one of the approaches in reliability-based design optimization which decouples the reliability analysis and optimization procedure. However, one of the challenges of the SORAs including search of the most probable point (MPP) is that it may not be able to converge if there are multiple MPPs[1]. Moreover, in practice one would need to calculate structural reliability with a set of random samples and alleviate the influence caused by the sampling size.

This study presents a quantile-based SORA for simultaneous shape and topology optimization of plane frames. The reliability constraint is expressed in terms of quantile which is estimated using the maximum entropy method (MEM) subjected to constraints on sample L-moments[2]. The shape of frame structure is determined by a set of force densities of the auxiliary truss that is irrelevant to the true loading and boundary conditions of the frame to be optimized[3].

2 Reliability-based shape and topology optimization of plane frames using force density method

Let $\boldsymbol{x}_{\text{free}}$, $\boldsymbol{y}_{\text{free}}$, and $\boldsymbol{x}_{\text{fix}}$, $\boldsymbol{y}_{\text{fix}}$ denote the x- and y-coordinates of free nodes and fixed nodes, respectively, of the frame to be optimized, and $\boldsymbol{t}$ be the force density vector applied on the members of auxiliary truss. The uncertainties are assumed to be random perturbations on $\boldsymbol{x}_{\text{free}}$, $\boldsymbol{y}_{\text{free}}$, $\boldsymbol{A}$ and Young's modulus $\boldsymbol{E}$ which are denoted by $\Delta\boldsymbol{x}_{\text{free}}$, $\Delta\boldsymbol{y}_{\text{free}}$, $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{E}$, respectively. The vectors of design and random variables are written as $\boldsymbol{d}=(\boldsymbol{x}_{\text{free}}(\boldsymbol{t}), \boldsymbol{y}_{\text{free}}(\boldsymbol{t}), \boldsymbol{A})$ and $\boldsymbol{\theta}=(\Delta\boldsymbol{x}_{\text{free}}, \Delta\boldsymbol{y}_{\text{free}}, \Delta\boldsymbol{A}, \Delta\boldsymbol{E})$, respectively. The reliability-based shape and topology optimization problem is formulated as

$$\begin{aligned}&\text{Miminize } W(\boldsymbol{d})\\ &\text{subject to } \Pr\{g_j(\boldsymbol{d};\boldsymbol{\theta})\le \bar{g}_j\}\ge R_j,\ j=1,2,\cdots,n;\\ &\qquad \underline{\boldsymbol{t}}\le \boldsymbol{t}\le \bar{\boldsymbol{t}};\ \underline{\boldsymbol{A}}\le \boldsymbol{A}\le \bar{\boldsymbol{A}}\end{aligned} \tag{1}$$

where $W(\boldsymbol{d})$ is the structural volume to be minimized; $g_j(\boldsymbol{d};\boldsymbol{\theta})$ is the jth performance function of nodal displacement and n is the number of performance functions; $\bar{g}_j$ is the prescribed upper bound of $g_j(\boldsymbol{d};\boldsymbol{\theta})$ and R_j is the target probability of $g_j(\boldsymbol{d};\boldsymbol{\theta})$ not to exceed $\bar{g}_j$. The lower and upper bars of $\boldsymbol{t}$ and $\boldsymbol{A}$ represent the lower and upper bounds for $\boldsymbol{t}$ and $\boldsymbol{A}$, respectively.

3 Quantile-based SORA

Based on the equivalent description between the reliability constraint and the quantile of structural response, the $(k+1)$th $(k\ge 0)$ deterministic optimization problem of quantile-based SORA of problem (2) is then formulated as

$$\begin{aligned}&\text{Miminize } W(\boldsymbol{d})\\ &\text{subject to } Q_{R_j}(\boldsymbol{d};\boldsymbol{\theta})\le \bar{g}_j-\bar{c}_j^{k+1},\ j=1,2,\cdots,n;\\ &\qquad \underline{\boldsymbol{t}}\le \boldsymbol{t}\le \bar{\boldsymbol{t}};\underline{\boldsymbol{A}}\le \boldsymbol{A}\le \bar{\boldsymbol{A}}\end{aligned} \tag{2}$$

where $Q_{R_j}(\boldsymbol{d};\boldsymbol{\theta})$ is the quantile of $g_j(\boldsymbol{d};\boldsymbol{\theta})$ with target probability R_j; $\bar{c}_j^{k+1}$ is the shifting value of $\bar{g}_j$ at the $(k+1)$th iteration which is calculated as

$$\bar{c}_j^{k+1}=Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta})-g_j(\boldsymbol{d}^k),\ j=1,2,\cdots,n \tag{3}$$

where $\boldsymbol{d}^k=(\boldsymbol{x}_{\text{free}}(\boldsymbol{t}^k), \boldsymbol{y}_{\text{free}}(\boldsymbol{t}^k), \boldsymbol{A}^k)$ is the solution of problem (2) at the kth iteration, and $Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta})$ represents the corresponding quantile.

4 Estimation of quantile

One of the main steps for solving problem (2) is to estimate the quantile $Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta})$ in Eq. (3). Suppose after the kth iteration that $g_j(\boldsymbol{d}^k;\boldsymbol{\theta})$ $(j=1,2,\cdots,n)$ is a continuous random variable $Z_j^k=g_j(\boldsymbol{d}^k;\boldsymbol{\theta})$, and let $Q_j^k(q)$ and $Q'^k_j(q)$ denote the quantile function of Z_j^k and its corresponding derivative for $0<q<1$, respectively. Let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_m$ denote the m sample vectors of uncertainty, and denote the corresponding m values of Z_j^k are by of $Z^k_{j,1}=g_j(\boldsymbol{d}^k;\boldsymbol{\theta}_1)$, $Z^k_{j,2}=g_j(\boldsymbol{d}^k;\boldsymbol{\theta}_2)$, …, $Z^k_{j,m}=g_j(\boldsymbol{d}^k;\boldsymbol{\theta}_m)$. By further defining $Z^k_{j,1:m}$, $Z^k_{j,2:m}$ …, $Z^k_{j,m:m}$ as the permutation of $Z^k_{j,1}$, $Z^k_{j,2}$, …, $Z^k_{j,m}$ in an ascending order, the rth order sample L-moment $l^k_{j,r}$ of Z_j^k is calculated as[3]

$$l^k_{j,r}=\sum_{s=0}^{r-1}(-1)^{r-1-s}\binom{r-1}{s}\binom{r-1+s}{s}b_s \tag{4}$$

where b_s is the sample probability weighted moments defined in Ref. [3]. According to Ref. [3], the estimation of $Q'^k_j(q)$ can be obtained using MEM subjected to the first n_{L} L-moment constraints, and the problem reads

$$\begin{aligned}&\text{Maximize } \int_0^1 \ln Q'^k_j(q)\,dq\\ &\text{subject to } h^k_{j,r}=\int_0^1 K_r(q)Q'^k_j(q)\,dq,\ r=1,2,\cdots,n_{\text{L}}\end{aligned} \tag{5}$$

where $h^k_{j,r}=l^k_{j,r}+[K_r(q)Q^k_j(q)]_0^1$, and $K_r(q)$ is a polynomial of q which is given in Ref. [3]. By using the method of Lagrange multipliers, $Q'^k_j(q)$ in Eq. (5) is given by the following form:

$$Q_j'^k(q) = 1 \Big/ \sum_{r=1}^{n_L} \lambda_{j,r} K_r(q) \quad (6)$$

where $\lambda_{j,r}$ (r=1,2,…,n_L) represents the unknown Lagrangian multiplier, which is determined by finding the stationary point of the following functional

$$\Gamma(\boldsymbol{\lambda}_j) = -\int_0^1 \ln\left(\sum_{r=1}^{n_L} \lambda_{j,r} K_r(q)\right) dq + \sum_{r=1}^{n_L} \lambda_{j,r} h_{j,r}^k \quad (7)$$

Once the values of λ_j and $Q'^k_j(q)$ in Eq. (6) are determined, the quantile function $Q_j^k(q)$ can be approximated by

$$Q_j^k(q) \approx Z_{j,1:m}^k + \int_0^u Q_j'^k(q) du \quad (8)$$

5 Example

A cantilever frame is optimized to verify the proposed method. The initial ground structure is shown in Fig. 1 which is pin-supported at nodes 1, 2 and 3, and a downward vertical load F=1000kN is applied at node 11; therefore, the fixed nodes for FDM are selected as nodes 1, 2, 3 and 11. Based on problem (3), the structural volume W is minimized such that the 0.99 quantile of downward vertical displacement of node 11, denoted by Q_{R11} is less than 3×10^{-3}m. The entries for the nominal Young's modulus $\boldsymbol{E}$ is 3×10^{11}Pa, and the uncertainties are given as follows

$$\begin{aligned} &\Delta \boldsymbol{x}_{\text{free}} \in [-0.02\boldsymbol{I}, 0.02\boldsymbol{I}];\ \Delta \boldsymbol{y}_{\text{free}} \in [-0.02\boldsymbol{I}, 0.02\boldsymbol{I}] \\ &\Delta \boldsymbol{A} \in [-0.02\boldsymbol{A}, 0.02\boldsymbol{A}];\ \Delta \boldsymbol{E} \in [-0.02\boldsymbol{E}, 0.02\boldsymbol{E}]; \end{aligned} \quad (9)$$

where $\boldsymbol{I}$ is the 8-by-1 identity vector.

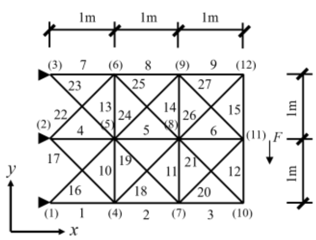
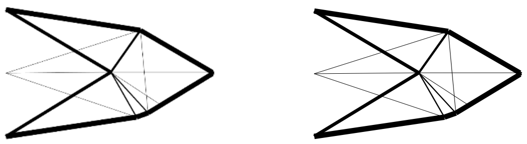


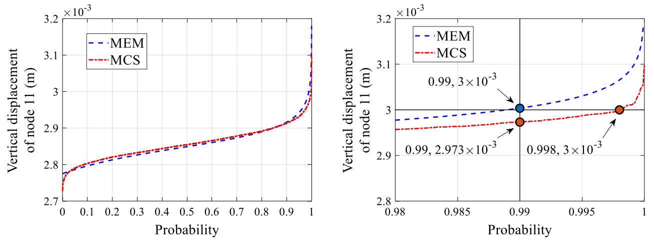
Fig.1 Initial ground structure of cantilever frame



(a) (b)

Fig. 2 Results at: (a) initial iteration; (b) final iteration

The first four sample L-moments are used to estimate Q_{R11} with sample size 50. The optimization procedure converged at k=3, and the results at the first and final iterations are shown in Fig. 2. The structural volume and quantile Q_{R11} of the results at initial and final iterations are listed in Table 1, where the quantiles obtained by MCS with sample size 1×10^5 are listed in the parentheses. Moreover, Fig.3 shows the quantile functions of vertical displacement of node 11 obtained by MCS and the proposed method for both results at initial and final iterations.



(a) (b)

Fig. 3 Q_{R11} at: (a) initial iteration; (b) final iteration

Table 1 Structural volume and quantile Q_{R11} of the results at the initial and final iterations

Result	Initial iteration	Final iteration
W (m^3)	9.2167×10^{-2}	9.8017×10^{-2}
Q_{R11}(m)	3.199×10^{-3} (3.175×10^{-3})	3.0×10^{-3} (2.973×10^{-3})

It can be seen from Figs. 2 and 3 and Table 1 that for the result at the initial iteration the probability of vertical displacement of node 11 exceeding the upper bound 3×10^{-3}m is about 0.8, while the reliability constraint on the vertical displacement of node 11 is satisfied for the result at the final iteration with the increase of 7% of structural volume compared to that at the initial iteration.

Conclusion

A quantile-based SORA method has been proposed for shape and topology optimization of plane frames. The shifting value on the bound of the constraint function is calculated in terms of quantile. The MEM is integrated to estimate the quantile function of the constraint using sample L-moments due to its less variability to the sample size. The FDM is introduced for shape optimization to alleviate the difficulty caused by the existence of extremely short members. The effectiveness of the method is demonstrated through an example of a cantilever frame to obtain the optimal solution which satisfies the reliability constraint.

Reference

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