2021年度日本建築学会大会（東海）

Sequential optimization and reliability assessment for shape and topology of plane frames using L-moments

Wei Shen[1], Makoto Ohsaki[1], Makoto Yamakawa[2]

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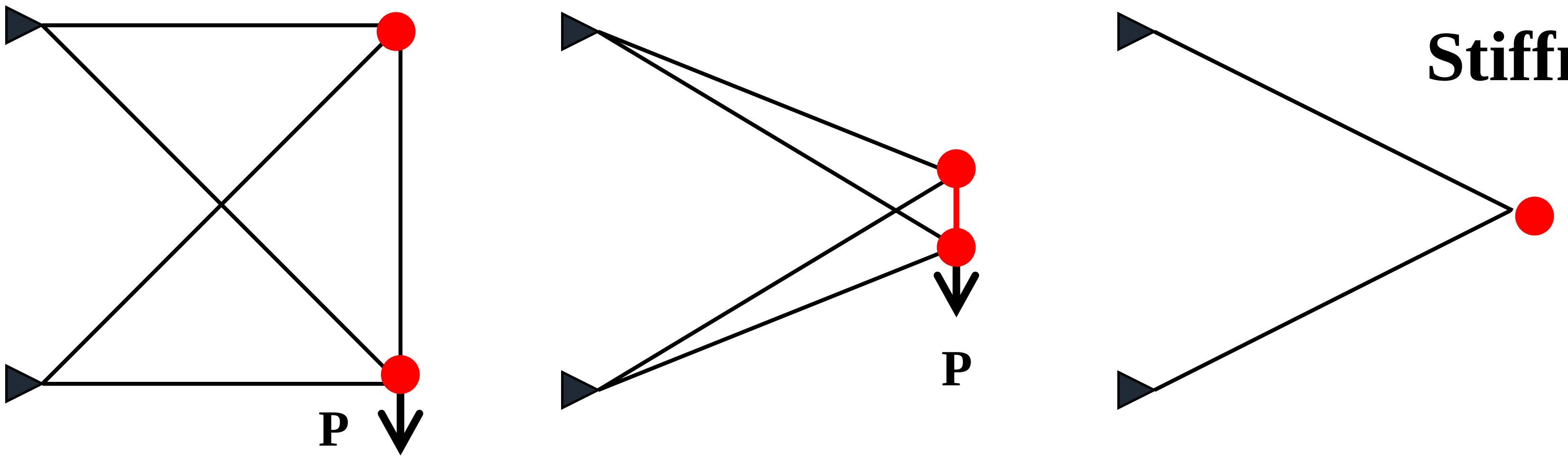
Existing Challenges

- **(1) Melting nodes of shape optimization (重複節点)**



Stiffness Matrix 剛性行列

Red nodes get close

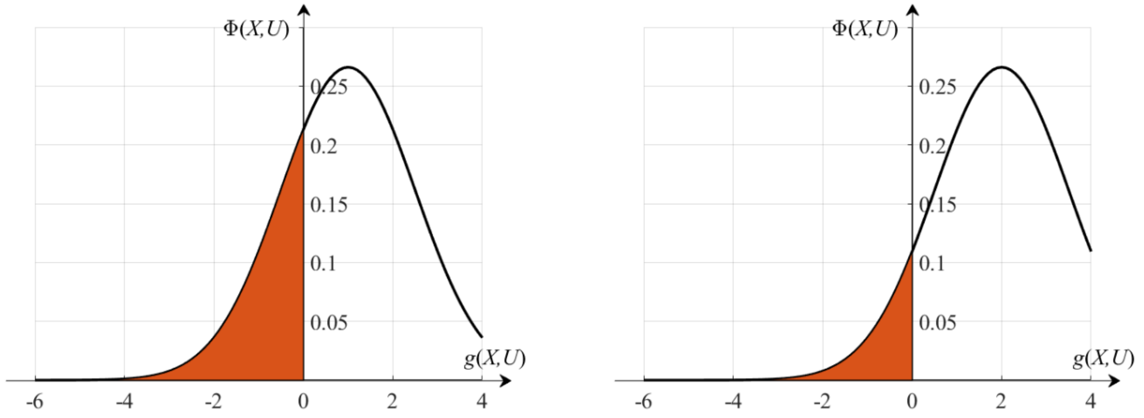
Singular 特異行列

- **(2) Reliability-based structural optimization (RBSO)**

$$\text{Miminize } W(\boldsymbol{d})$$

$$\text{subject to } \Pr\left\{g_j(\boldsymbol{d};\boldsymbol{\theta}) \le \bar{g}_j\right\} \ge R_j,\ j=1,2,\cdots,n;$$

信頼性に基づく構造最適化



RBSO of plane frame

RBSO of shape and topology optimization of plane frame

$$\text{Miminize } W(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{A})$$

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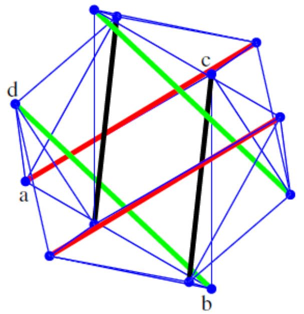
$\boldsymbol{\theta}$: Random parameter (不確定性); R_j: Target probability (性能制約を満たす確率)

To prevent the existence of short member 軸力密度法



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Force density 軸力密度

Axial force 軸力

$$t = \frac{N}{L}$$

Member length 部材長さ

diagonal matrix (対角行列) of $\boldsymbol{t}$

$$\boldsymbol{x}_{\text{free}} = -\left(\tilde{\boldsymbol{T}}_{\text{free}}^{T} \text{diag}(\boldsymbol{t}) \tilde{\boldsymbol{T}}_{\text{free}}\right)^{-1} \tilde{\boldsymbol{T}}_{\text{free}}^{T} \text{diag}(\boldsymbol{t}) \tilde{\boldsymbol{T}}_{\text{fix}} \boldsymbol{x}_{\text{fix}}$$

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Connectivity matrix (接続行列)

Quantile-based RBSO

- **(1) Problem statement**

確率に関する制約

$$\text{Miminize } W(\boldsymbol{d})$$

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With $\boldsymbol{d}$ = ($x_{\text{free}}(t)$, $y_{\text{free}}(t)$, A)

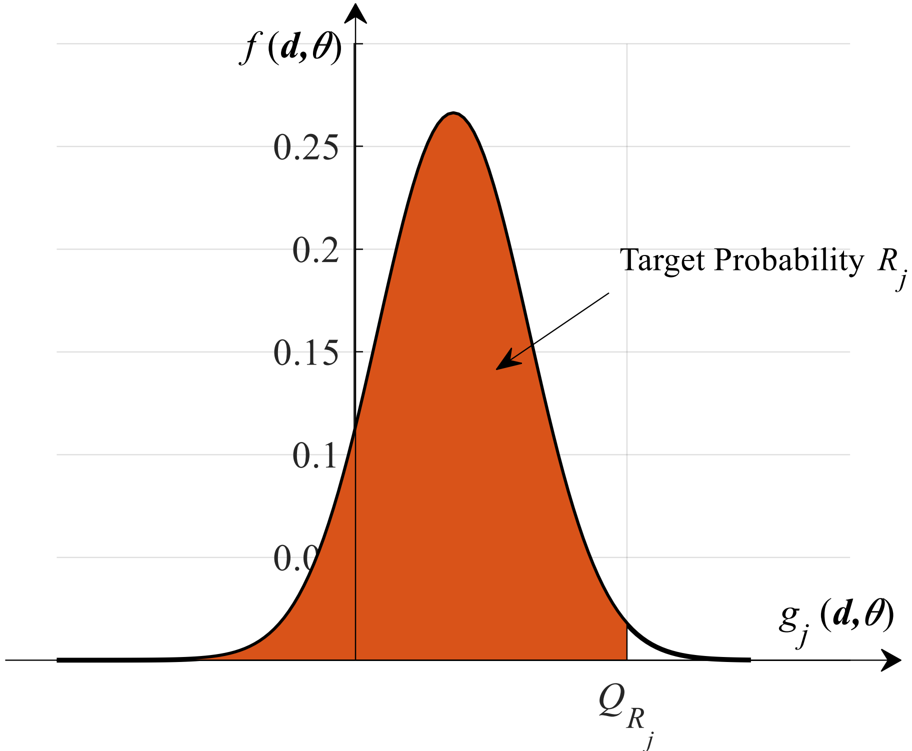
Rewrite the problem using quantile (分位数)

$$\text{Miminize } W(\boldsymbol{d})$$

$$\text{subject to } Q_{R_j}(\boldsymbol{d};\boldsymbol{\theta}) \le \bar{g}_j,\ j=1,2,\cdots,n;$$

With definition of quantile (分位数定義)

$$Q_{R_j}(\boldsymbol{d};\boldsymbol{\theta}) = \inf\left\{Q : \Pr\left\{g_j(\boldsymbol{d};\boldsymbol{\theta}) \le Q\right\} \ge R_j\right\},\ j=1,2,\cdots,n$$



Problem formulation

- **Sequential optimization and reliability assessment (SORA)**

Du XP, Chen W (2004)

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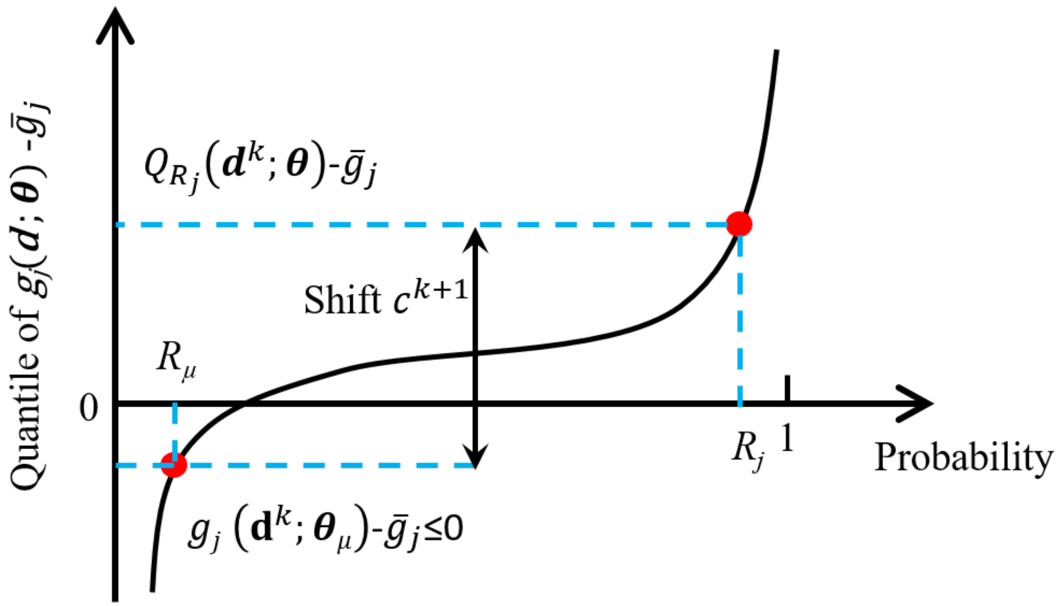
$$\text{subject to } Q_{R_j}(\boldsymbol{d};\boldsymbol{\theta}) \le \bar{g}_j,\ j=1,2,\cdots,n;$$

信頼性制約を確定制約に変換し、確定的な最適化問題を解くことを繰り返す

$$\text{Miminize } W(\boldsymbol{d})$$
$$\text{subject to } g_j(\boldsymbol{d}) \le \bar{g}_j - \bar{c}_j^{k+1},\ j=1,2,\cdots,n$$

$$\bar{c}_j^{k+1} = Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta}) - g_j(\boldsymbol{d}^k),\ j=1,2,\cdots,n$$

Decouple uncertainty from structural optimization



Estimation of quantile

Determine the desired quantile $Q_{R_j}\left(\boldsymbol{d}^k;\boldsymbol{\theta}\right)$ **: how?**

Recall that: $Q_{R_j}\left(\boldsymbol{d};\boldsymbol{\theta}\right)=\inf\left\{Q:\Pr\left\{g_j\left(\boldsymbol{d};\boldsymbol{\theta}\right)\le Q\right\}\ge R_j\right\},\ j=1,2,\cdots,n$

Consider $g_j\left(\boldsymbol{d};\boldsymbol{\theta}\right)$ **as a random variable** $Z_j^k=g_j\left(\boldsymbol{d}^k;\boldsymbol{\theta}\right)$ **確率変数となる**

$F_j^k\left(z_j^k\right)$ **Cumulative distribution function:累積分布関数**

Inverse function ⟷ **逆関数**

$Q_j^k\left(q\right)$ **Quantile function:分位関数**

$0\le q<1$

$f_j^k\left(z_j^k\right)=dF_j^k\left(z_j^k\right)/dz_j^k$ **Probability distribution function:確率密度関数**

$Q_j'^k\left(q\right)=dQ_j^k\left(q\right)/dq$ **Quantile density function**

Maximum entropy method

Entropy of random variable (確率変数のエントロピー)

$$H_j^k = \int_{-\infty}^{+\infty} \left\{ -\ln f_j^k\left(z_j^k\right) \right\} f_j^k\left(z_j^k\right) dz_j^k = \int_0^1 \ln Q_j'^k(q)\, dq$$

$$Q_j'^k(q) = \frac{1}{f_j^k\left(z_j^k\right)}$$ 逆数

To derive $Q_j'^k(q)$ using Maximum Entropy method

最大エントロピー法

Maximize $\int_0^1 \ln Q_j'^k(q)\, dq$

subject to $l_{j,r}^k = \int_0^1 K_r(q) Q_j'^k(q)\, dq - \left[K_r(q) Q_j^k(q) \right]_0^1, \quad r = 1, 2, \cdots, n_{\mathrm{L}}$

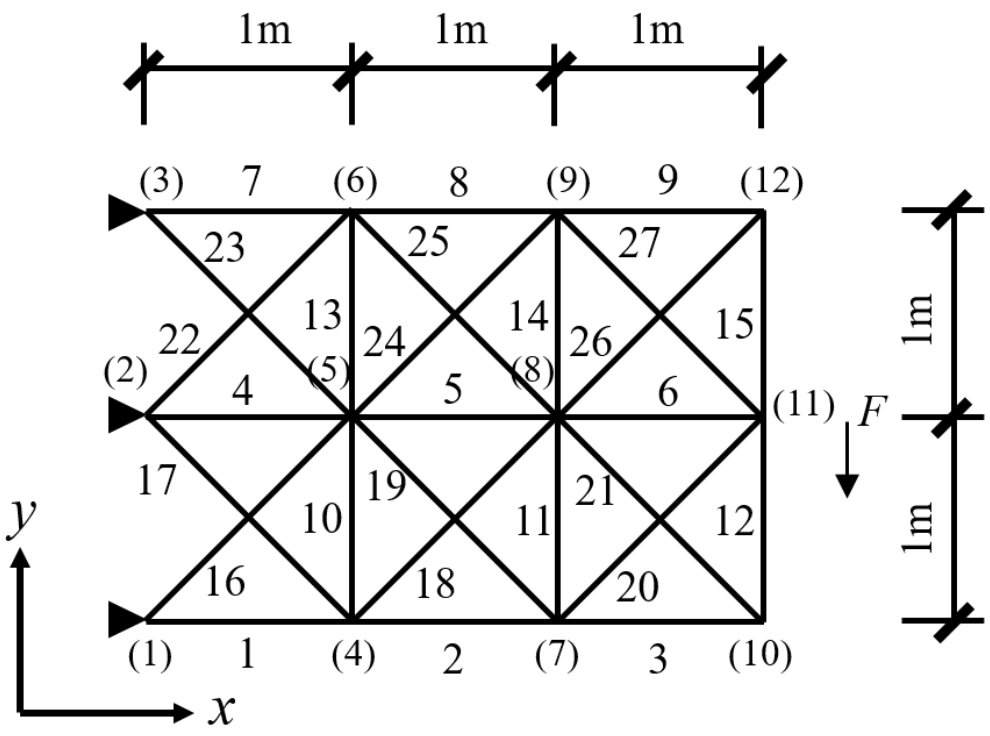
$l_{j,r}^k$: **Sample linear moment**
変量統計の線形モーメント

$$K_r(q) = \int_q^1 P_{r-1}^*(v)\, dv$$

P_{r-1}^*: **Shifted Legendre polynomial**
ずらしルジャンドル多項式

Numerical Example

➢ **Example:**



To minimize the structural volume with displacement constraint on Node 11

節点11の変位に対する信頼性制約下での平面骨組総体積を最小化する

$$\text{Miminize } W\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{t}), \boldsymbol{y}_{\text{free}}(\boldsymbol{t}), \boldsymbol{A}\right)$$

$$\text{subject to } Q_{R_{11}}\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{t}), \boldsymbol{y}_{\text{free}}(\boldsymbol{t}), \boldsymbol{A}\right) \leq 3\times10^{-3}\,\text{m};$$

$$\underline{\boldsymbol{t}} \leq \boldsymbol{t} \leq \overline{\boldsymbol{t}};\ \ \underline{\boldsymbol{A}} \leq \boldsymbol{A} \leq \overline{\boldsymbol{A}}$$

Target probability: R_{11}=0.99

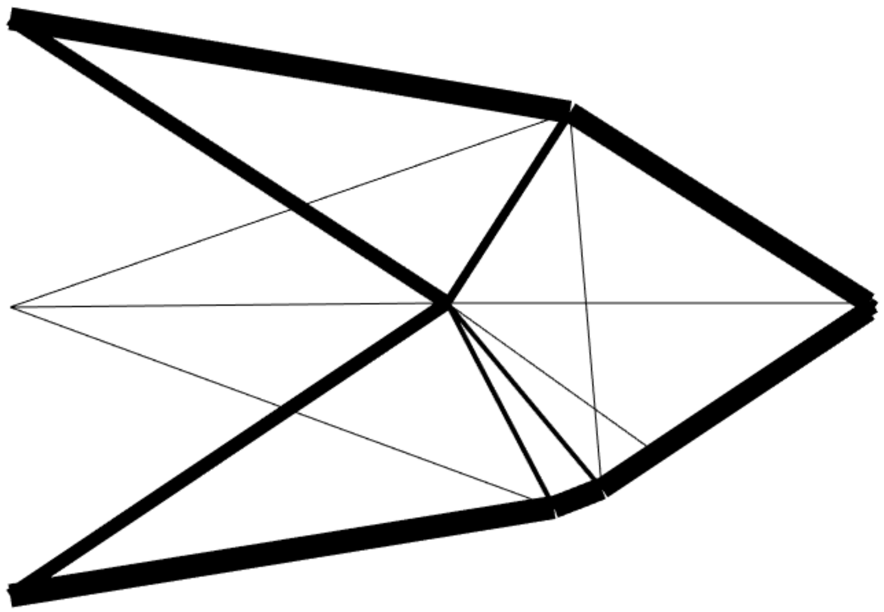
Uncertainty: $\boldsymbol{\theta} = \left(\Delta\boldsymbol{x}_{\text{free}}, \Delta\boldsymbol{y}_{\text{free}}, \Delta\boldsymbol{A}, \Delta\boldsymbol{E}\right)$

不確定性:節点位置、断面積、ヤング係数

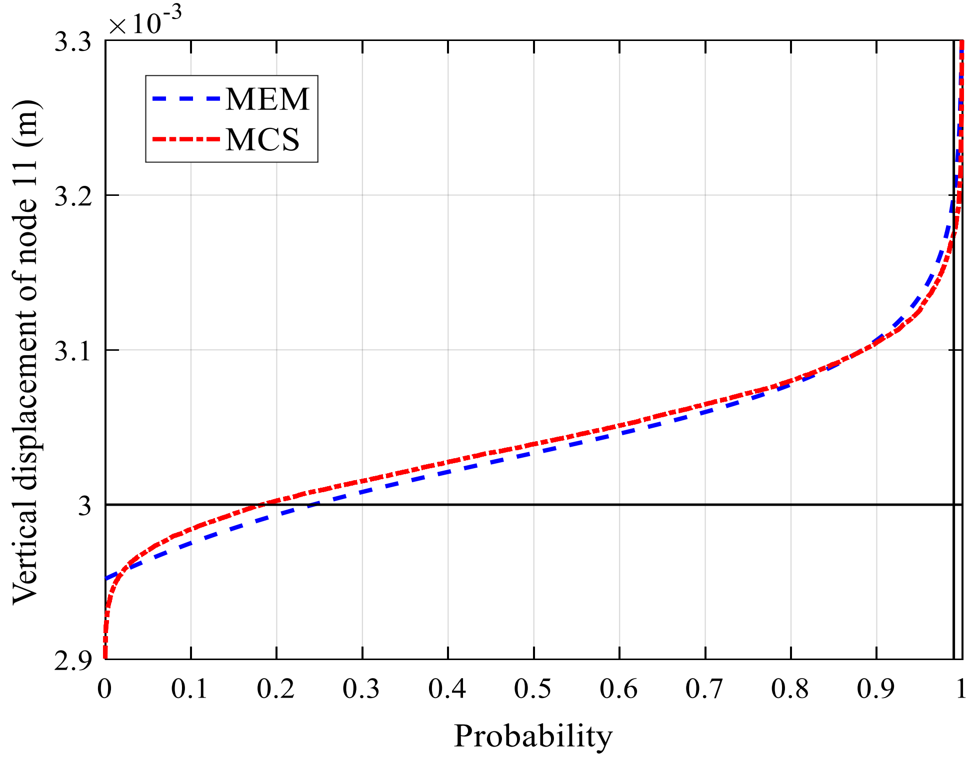
$Q_{R_{11}}$

Numerical Example

➢First iteration:



MCS: Monte Carlo simulation

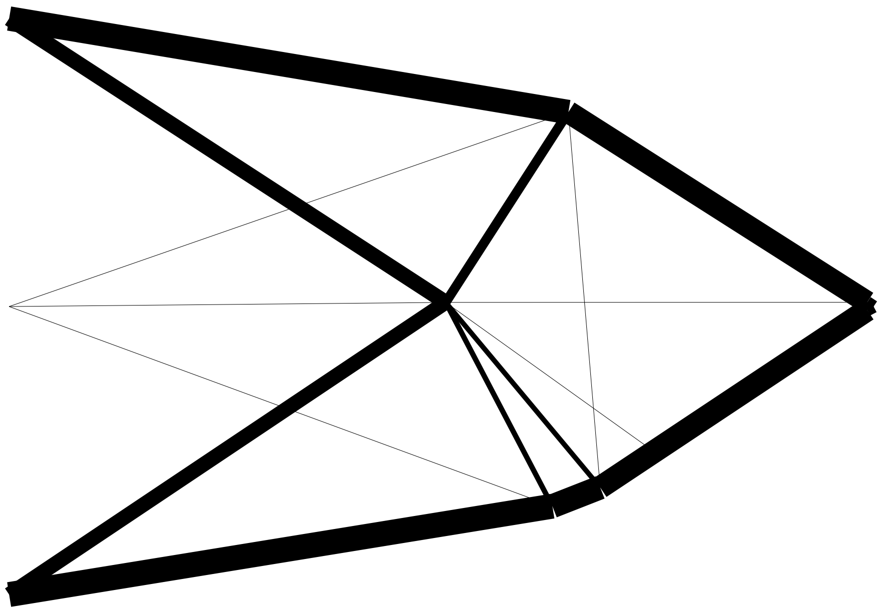


	Result	Initial iteration	Final iteration
総体積	Structural volume (m^3)	9.2167×10^{-2}	9.8017×10^{-2}
分位数	$Q_{R_{11}}$ (m)	3.199×10^{-3} (3.175×10^{-3})	3.0×10^{-3} (2.973×10^{-3})

(·): MCS

Numerical Example

➢Final iteration:



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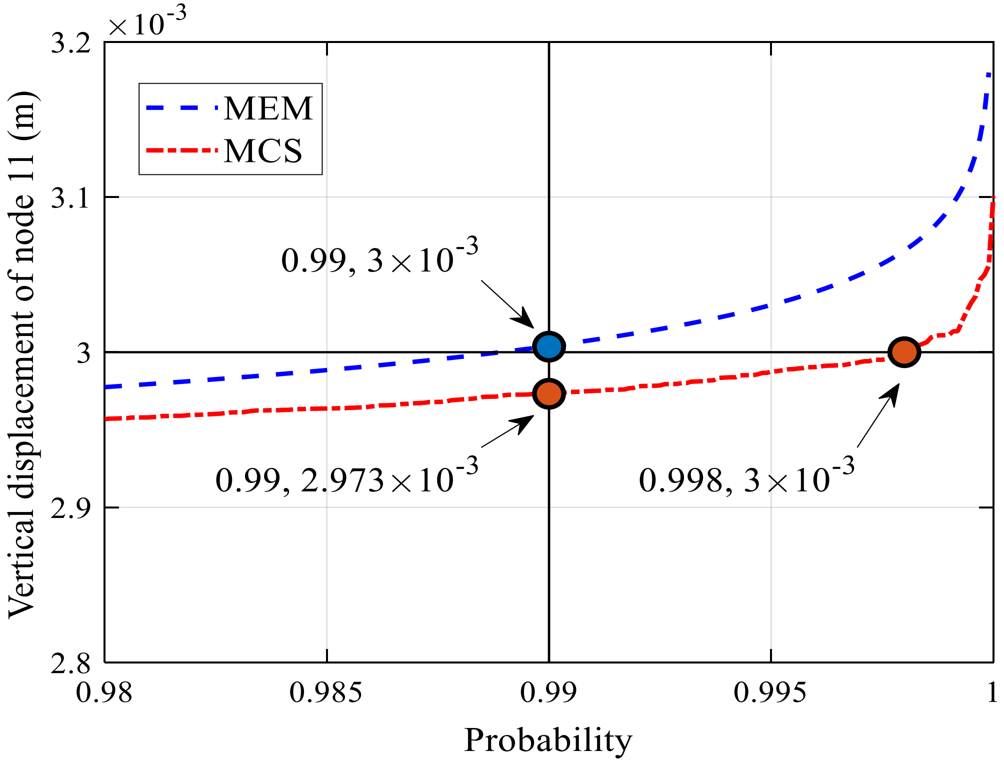

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Summary and conclusion

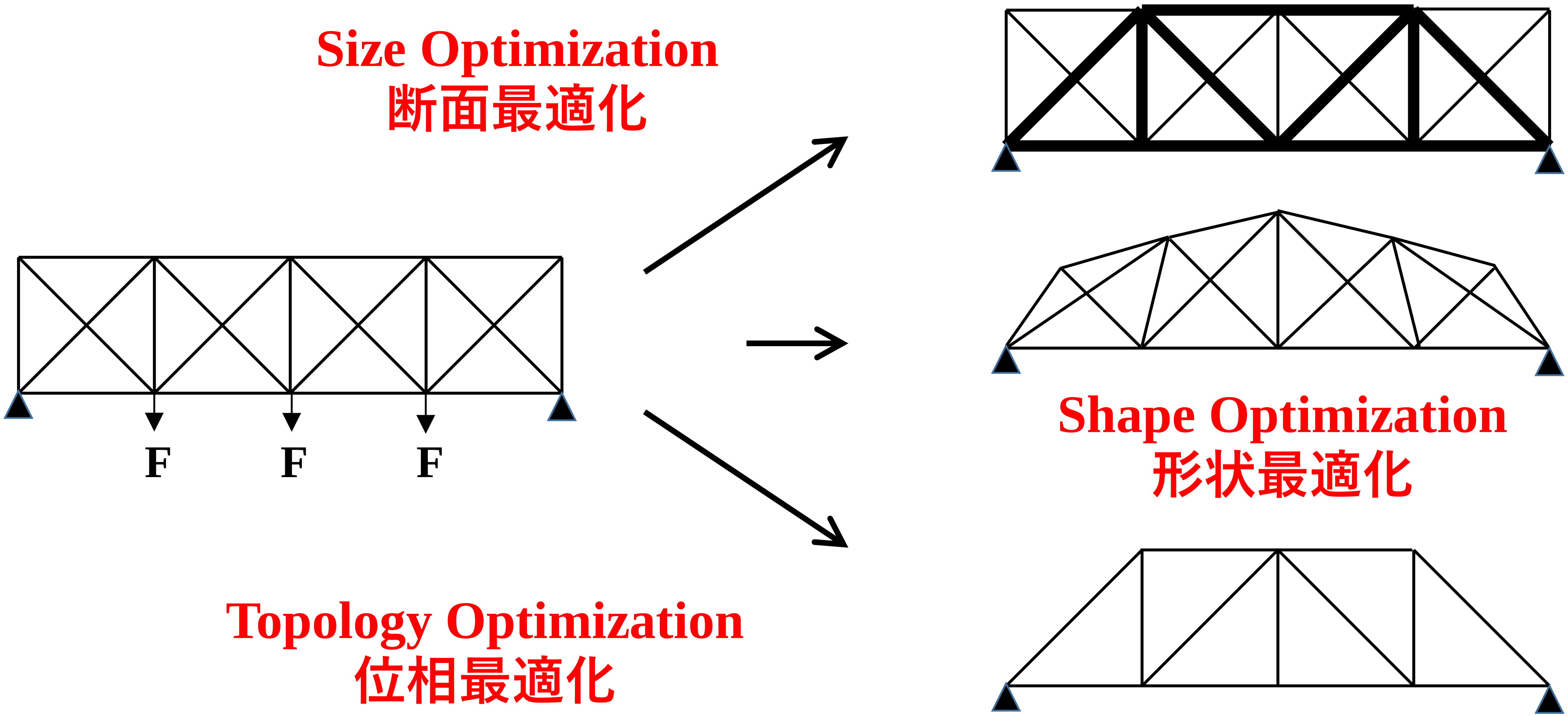
➢Brief Summary of the presentation:

- Reliability-based shape and topology optimization
- Quantile-based SORA
- Estimation of quantile function using sample L-moments
- Force density method

➢The proposed method has the following conclusions:

- Estimation of quantile function can be achieved
- A result satisfying the probability constraints can be found
- Melting nodes can be avoid

Optimization of Frames



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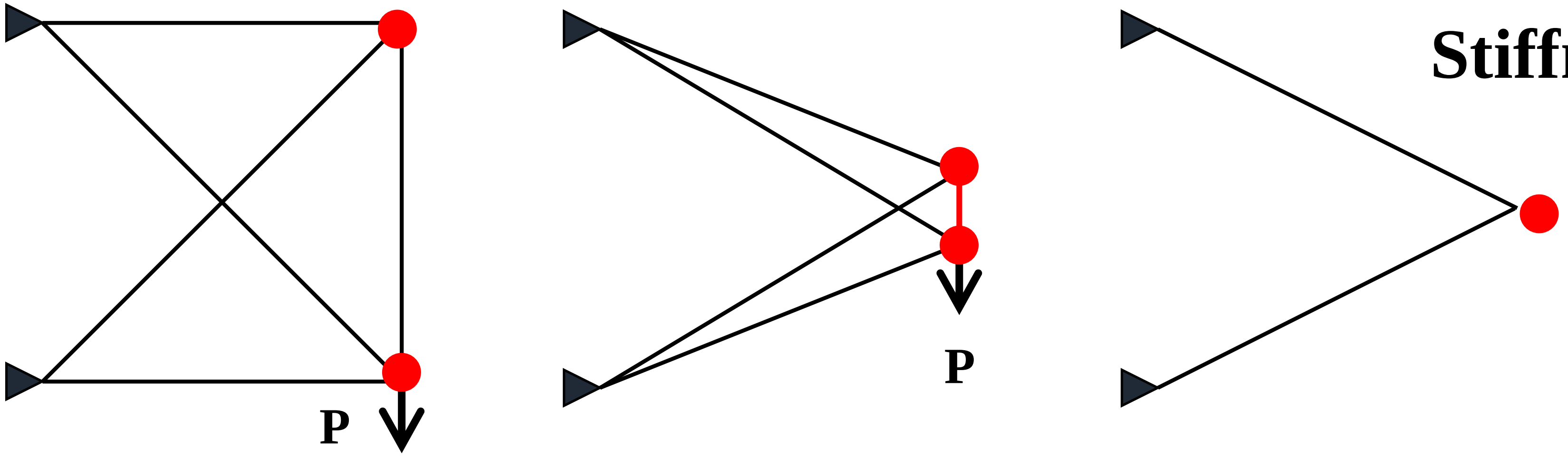

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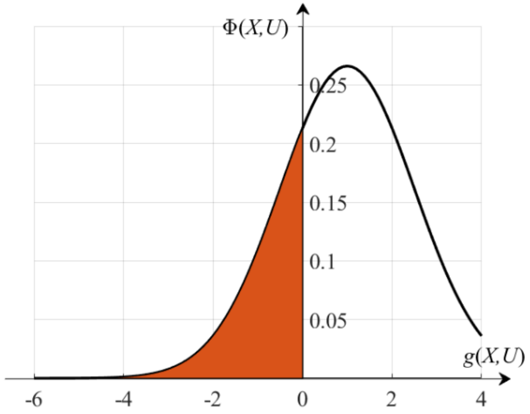

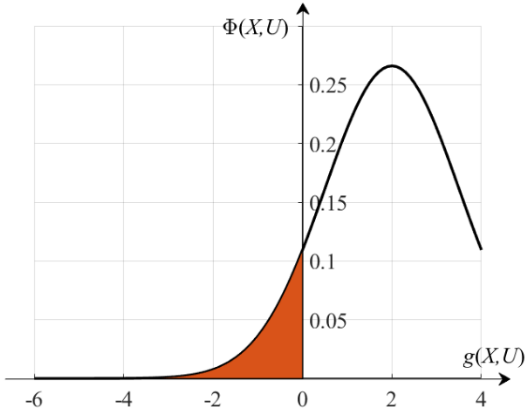

RBSO of plane frame

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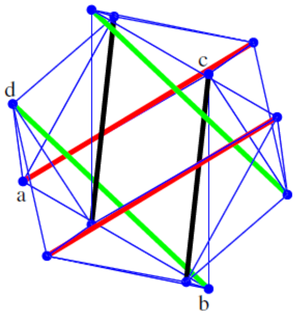

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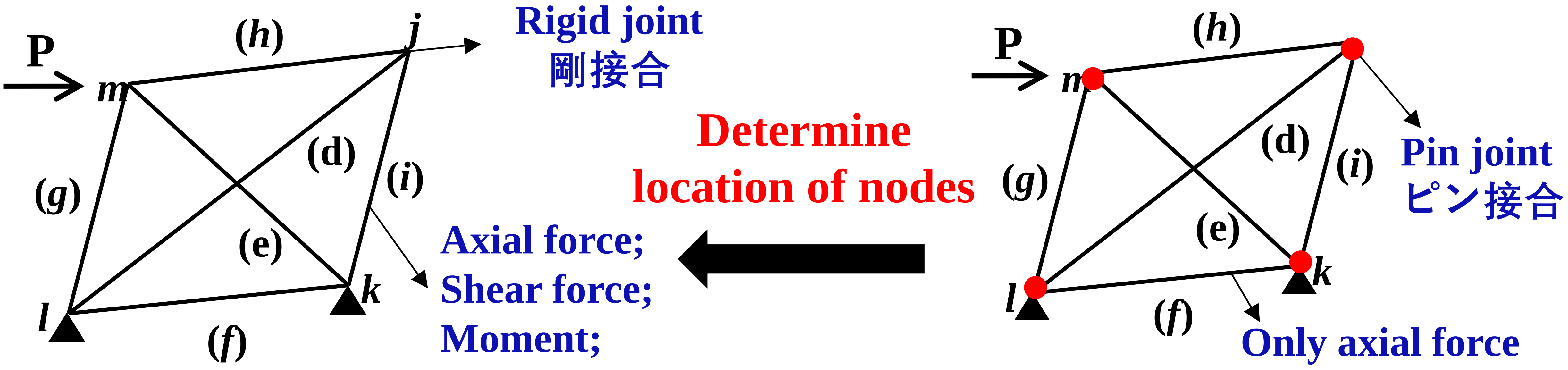
Frame structure **Auxiliary truss structure**



Design variable: *t*-force density;
***A*-cross-sectional area**

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Quantile-based RBDO

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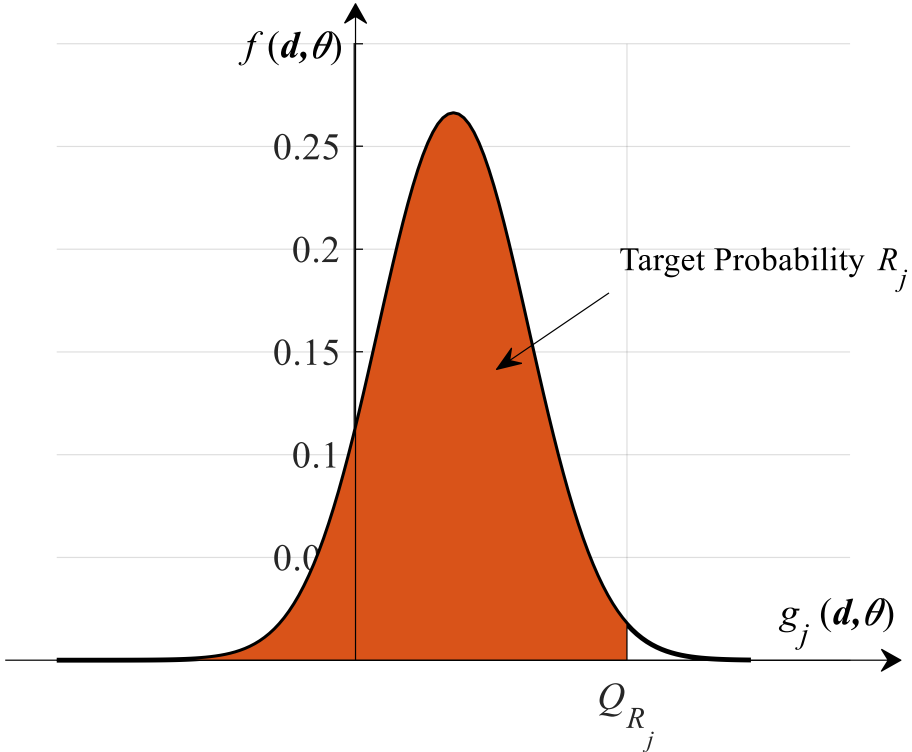

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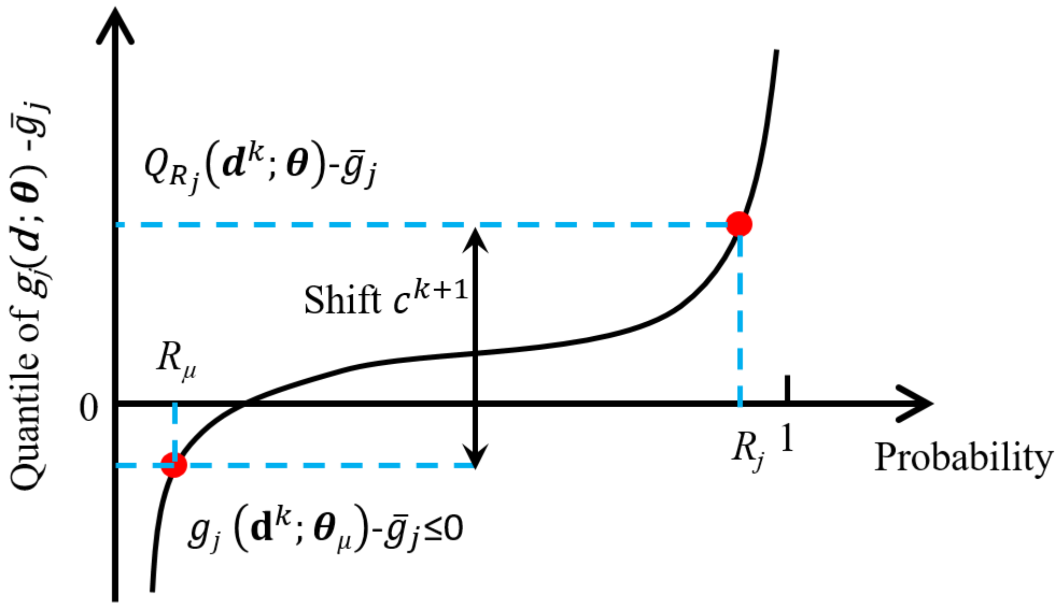

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Entropy of random variable (確率変数のエントロピー)

$$H_j^k = \int_{-\infty}^{+\infty} \left\{ -\ln f_j^k\left(z_j^k\right) \right\} f_j^k\left(z_j^k\right) dz_j^k = \int_0^1 \ln Q_j'^k(q)\, dq$$

$$Q_j'^k(q) = \frac{1}{f_j^k\left(z_j^k\right)}$$ 逆数

To derive $Q_j'^k(q)$ using Maximum Entropy method

最大エントロピー法

Maximize $\int_0^1 \ln Q_j'^k(q)\, dq$

subject to $l_{j,r}^k = \int_0^1 K_r(q) Q_j'^k(q)\, dq - \left[K_r(q) Q_j^k(q) \right]_0^1, \quad r = 1, 2, \cdots, n_L$

$l_{j,r}^k$: **Sample linear moment**
変量統計の線形モーメント

$$K_r(q) = \int_q^1 P_{r-1}^*(v)\, dv$$

$P_{r-1}^*(v)$: **Shifted Legendre polynomial**
ずらしルジャンドル多項式

Lagrangian method

Define: $h_{j,r}^{k}=l_{j,r}^{k}+\left[K_r(q)Q_j^k(q)\right]_0^1$

ラグランジュの未定乗数法

Solved by Lagrangian multiplier method

Maximize $\int_0^1 \ln Q_j'^k(q)\,dq$

subject to $h_{j,r}^{k}=\int_0^1 K_r(q)Q_j'^k(q)\,dq,\ r=1,2,\cdots,n_{\mathrm{L}}$

Lagrangian functional (ラグランジュ汎関数):

$$\bar{H}_j^k(q)=\int_0^1 \ln Q_j'^k(q)\,dq-\sum_{r=1}^{n_{\mathrm{L}}}\lambda_{j,r}\left(\int_0^1 K_r(q)Q_j'^k(q)\,dq-h_{j,r}^{k}\right)$$

Solution ➡

$$Q_j'^k(q)=\frac{1}{\sum_{r=1}^{n_{\mathrm{L}}}\lambda_{j,r}K_r(q)}$$

To determine the $\lambda_{j,r}$ we solve the following problem:

$$\mathrm{Min}:\Gamma(\lambda_j)=-\int_0^1 \ln\left(\sum_{r=1}^{n_{\mathrm{L}}}\lambda_{j,r}K_r(q)\right)dq+\sum_{r=1}^{n_{\mathrm{L}}}\lambda_{j,r}h_{j,r}^{k}$$

Unconstrained and convex

制約なし凸最適化

Quantile function

After obtaining the Lagrangian multiplier:

$$Q_j^k(q) = \boxed{Q_j^k(0)} + \int_0^u Q_j'^k(q)\,dq \qquad \textbf{with} \qquad Q_j'^k(q) = \frac{1}{\sum_{r=1}^{n_L} \lambda_{j,r} K_r(q)}$$

↓ **unknown, approximated**

最小順序統計量

$$Q_j^k(q) \approx \boxed{Z_{j,1:m}^k} + \int_0^u Q_j'^k(q)\,du$$

0からR_jまでの積分

For desired quantile: $Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta}) \approx Z_{j,1:m}^k + \int_0^{\boxed{R_j}} Q_j'^k(q)\,dq$

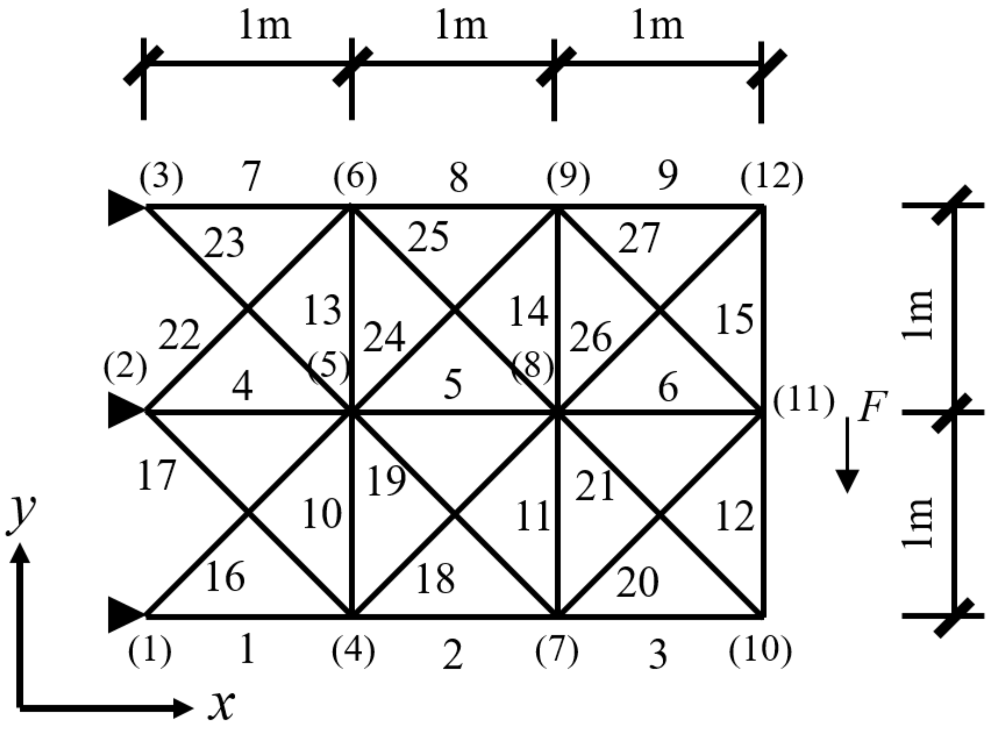
Back to SORA problem formulation:

Miminize $W(\boldsymbol{d})$

subject to $g_j(\boldsymbol{d}) \le \bar{g}_j - \boxed{\bar{c}_j^{k+1}}$, $j = 1, 2, \cdots, n$ $\qquad \boxed{\bar{c}_j^{k+1} = Q_{R_j}(\boldsymbol{d}^k;\boldsymbol{\theta}) - g_j(\boldsymbol{d}^k),\ j = 1, 2, \cdots, n}$

Numerical Example

- **Example:**



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$$\underline{\boldsymbol{t}} \leq \boldsymbol{t} \leq \overline{\boldsymbol{t}};\ \ \underline{\boldsymbol{A}} \leq \boldsymbol{A} \leq \overline{\boldsymbol{A}}$$

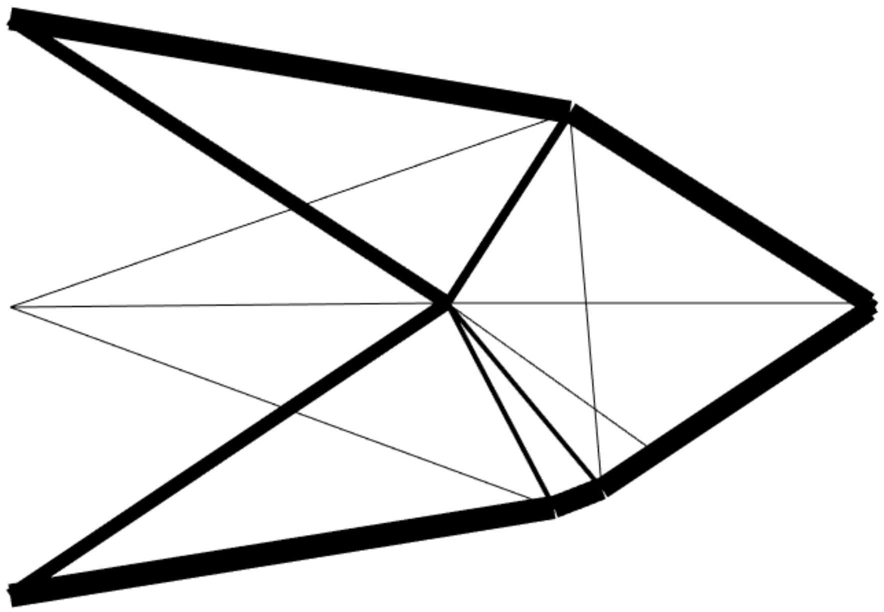
Target probability: R_{11}=0.99

Uncertainty: $\boldsymbol{\theta} = \left(\Delta\boldsymbol{x}_{\text{free}}, \Delta\boldsymbol{y}_{\text{free}}, \Delta\boldsymbol{A}, \Delta\boldsymbol{E}\right)$

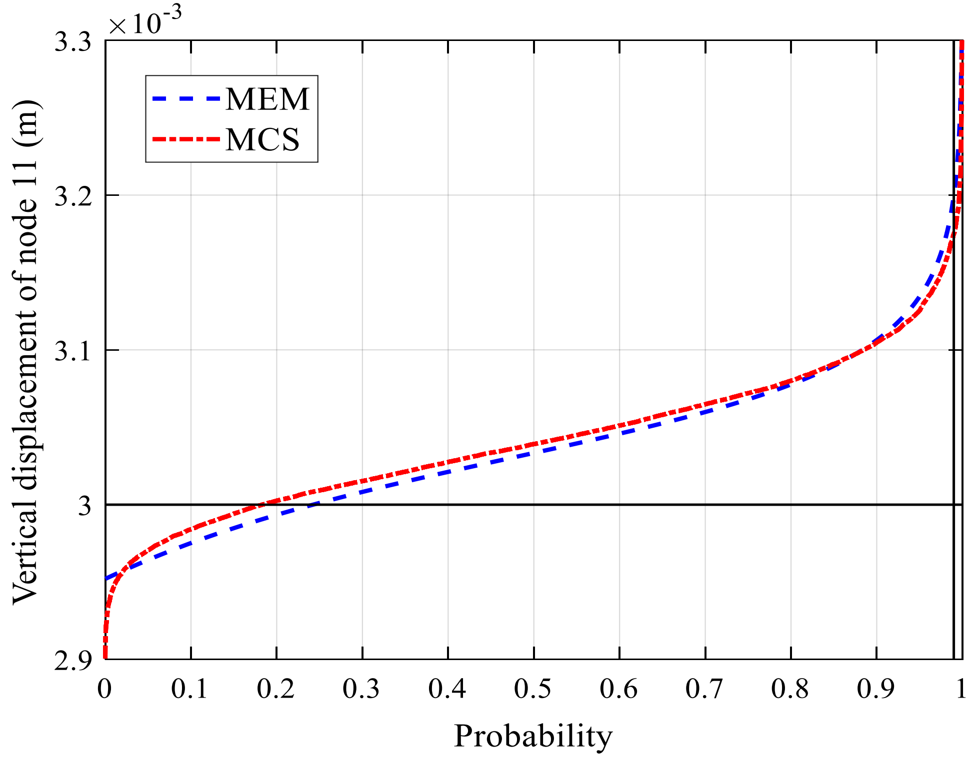
不確定性:節点位置、断面積、ヤング係数

Numerical Example

➢First iteration:



MCS: Monte Carlo simulation



	Result	Initial iteration	Final iteration
総体積	Structural volume (m^3)	9.2167×10^{-2}	9.8017×10^{-2}
分位数	$Q_{R_{11}}$ (m)	3.199×10^{-3} (3.175×10^{-3})	3.0×10^{-3} (2.973×10^{-3})

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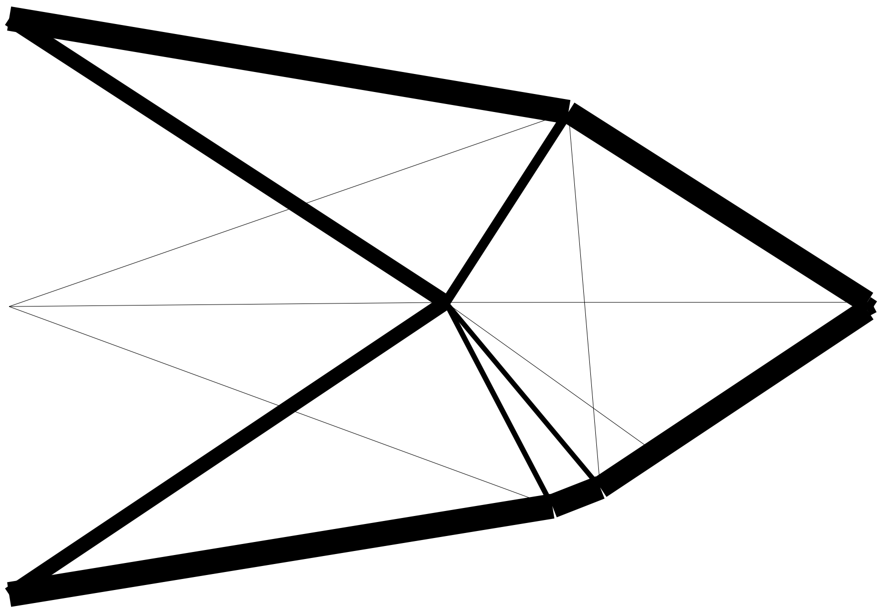
Numerical Example

➢Final iteration:



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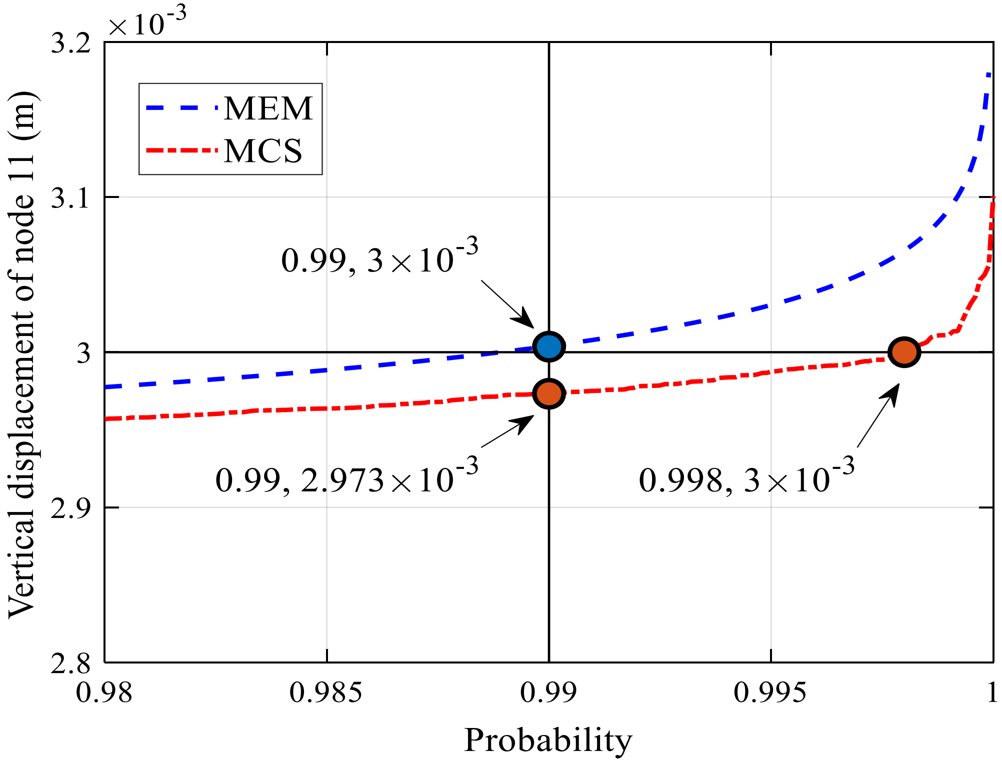

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Summary and conclusion

➢Brief Summary of the presentation：

- Reliability-based shape and topology optimization
- Quantile-based SORA
- Estimation of quantile function using sample L-moments
- Force density method

➢The proposed method has the following conclusions：

- Estimation of quantile function can be achieved
- A result satisfying the probability constraints can be found
- Melting nodes can be avoid

Thanks for your kind attention

ご清聴ありがとうございます