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# Shape and Topology Optimization of Plane Frame Structure Using Force Density Method

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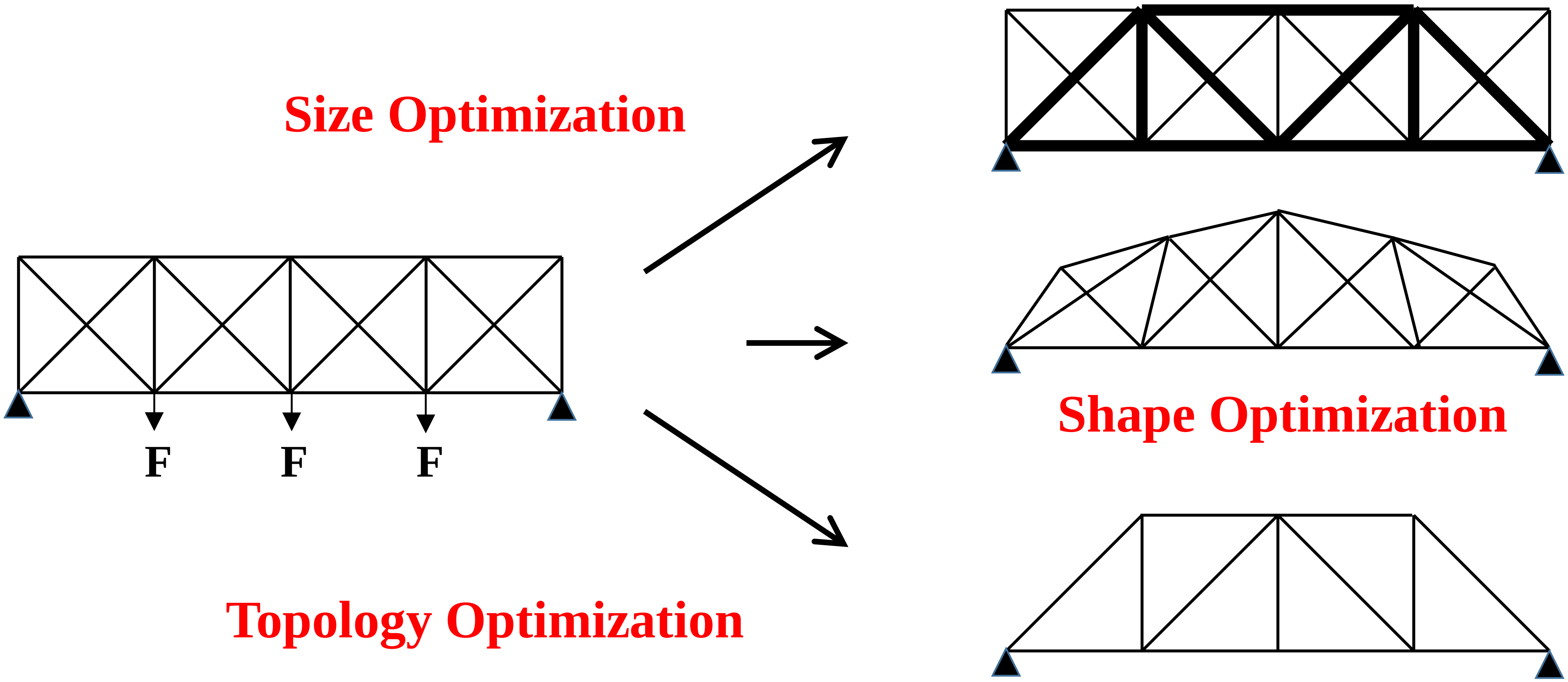
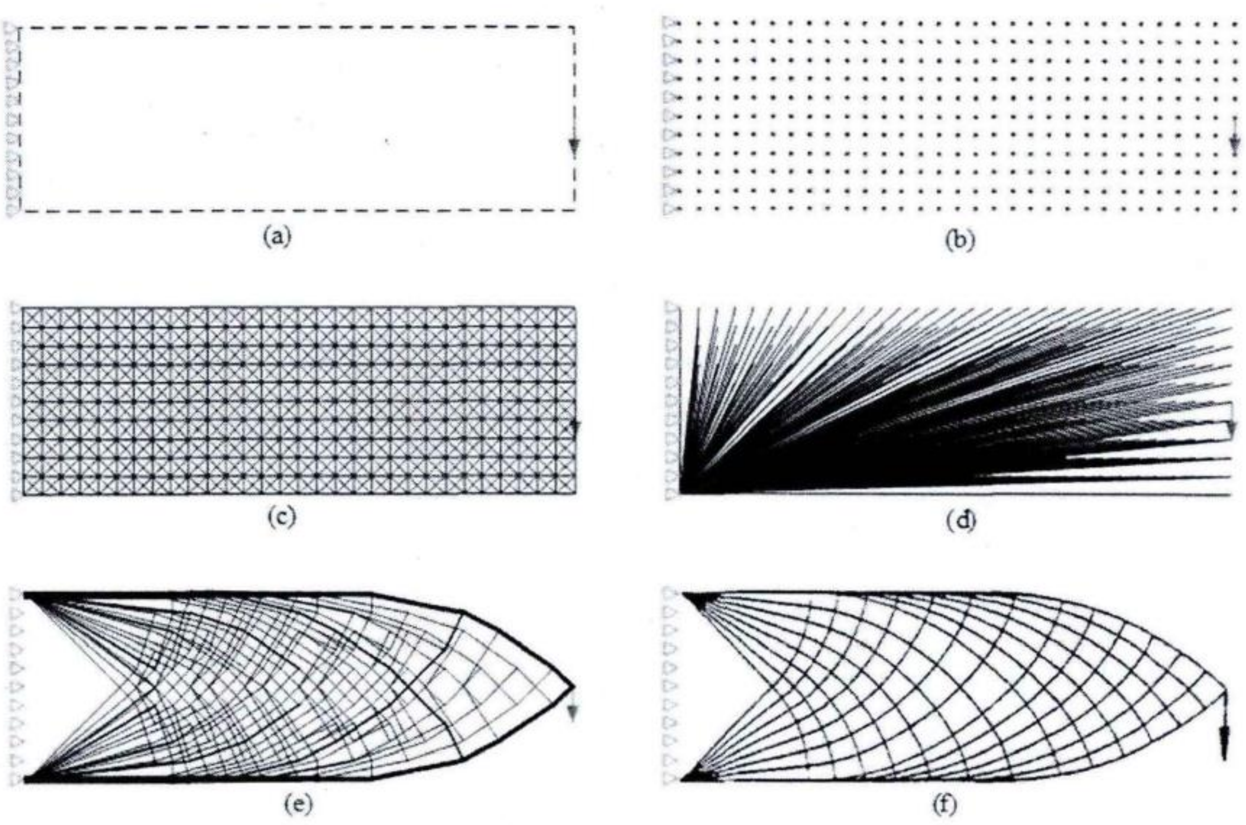
# *Optimization of Frames*



# *Characteristics* 1

- **Ground structure method**




**Topology optimization**

⬇

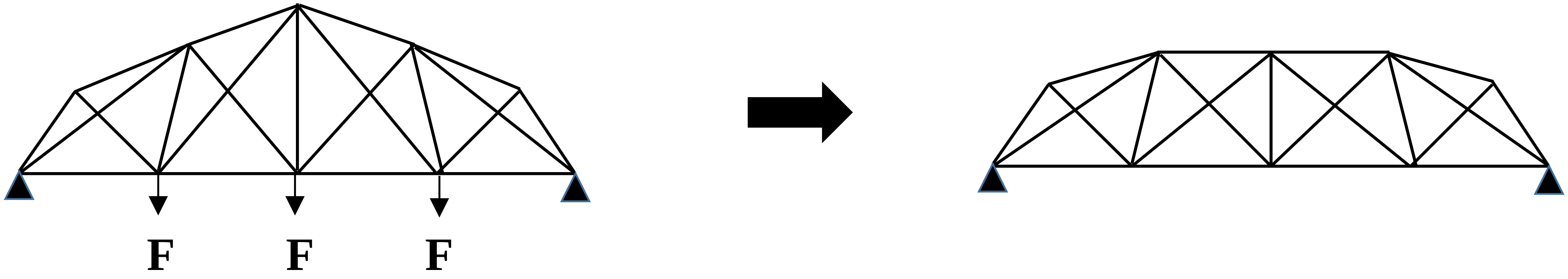
**Size optimization**

**Challenge: Influence of initial nodes and elements**

**Source**: Ge Gao, Research on theory and application of truss structure, 2016, PhD Thesis.

# *Possible approach*

- **Simultaneous Optimization**

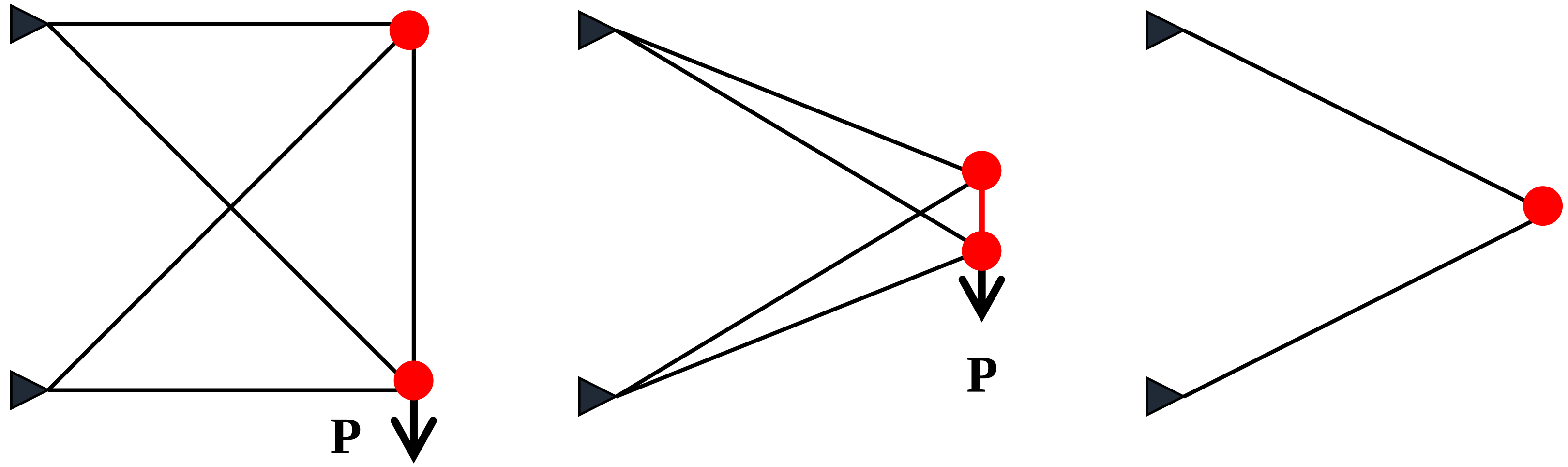



**Simultaneous optimization of shape and topology**

**Location of Nodes** ✚ **Cross-section area of element**

# *Characteristics 2*

- **Melting nodes**




**Stiffness Matrix**

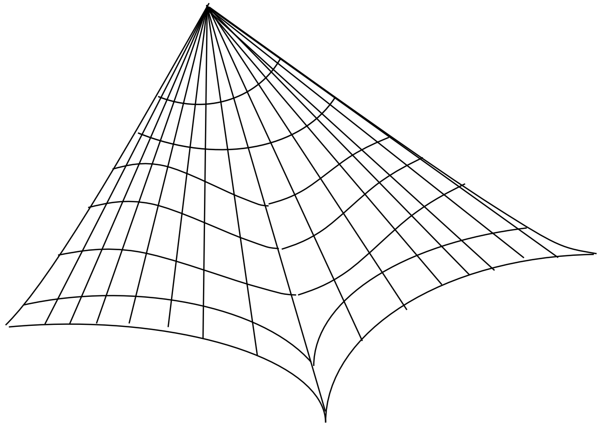
**Red nodes get close**

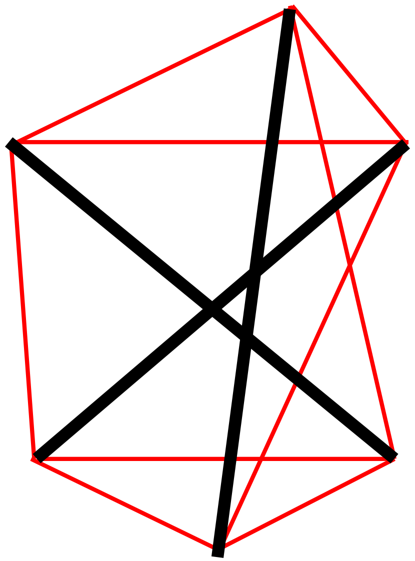
**Singular**

**Desirable to avoid existence of extremely short member**

# *Possible approach*

- **Force density method(FDM)**





**Widely used in tension and tensegrity structure**

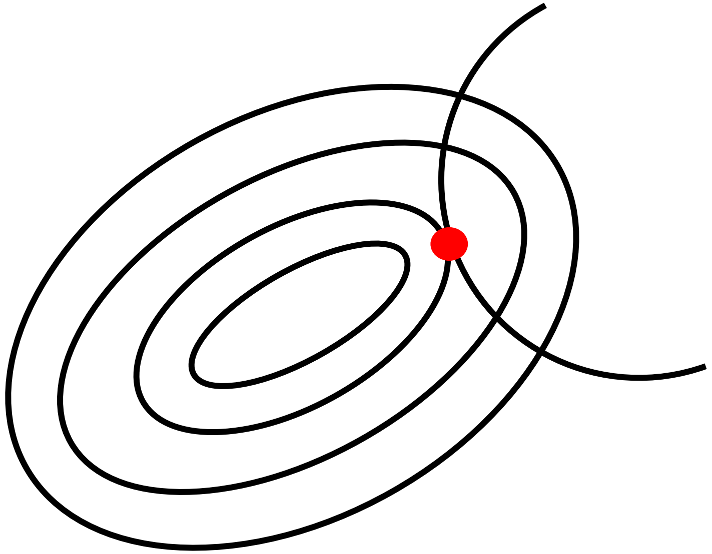
**Force density** $q = \frac{N}{L}$ **Axial force** / **Member length**

**Determine**

***q* for each element** → **Structural Shape**

# *Characteristic 3*

- **Nonlinear Programming**



$$\text{Min } f(\mathbf{x})$$
$$s.t.\ h(\mathbf{x}) = 0$$
$$g(\mathbf{x}) \leq 0$$

**Shape optimization → Nodal location**

↓ **Nonlinear**

**Structure Stiffness Matrix**

↓ **Sensitivity analysis**

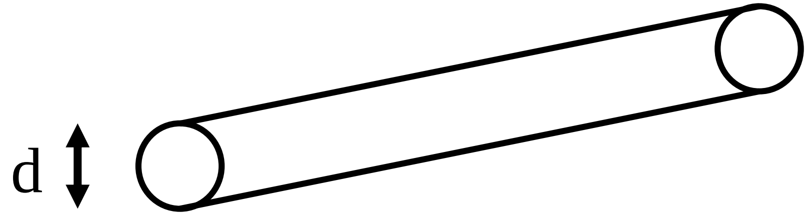
**Sequential quadratic programming (SQP)**

# Problem Formulation

**Minimize :** $\boldsymbol{U}^T \boldsymbol{K} \boldsymbol{U}$

**Subject to:** $V \leq V_{upper}$

**Euler-Bernoulli beam element**



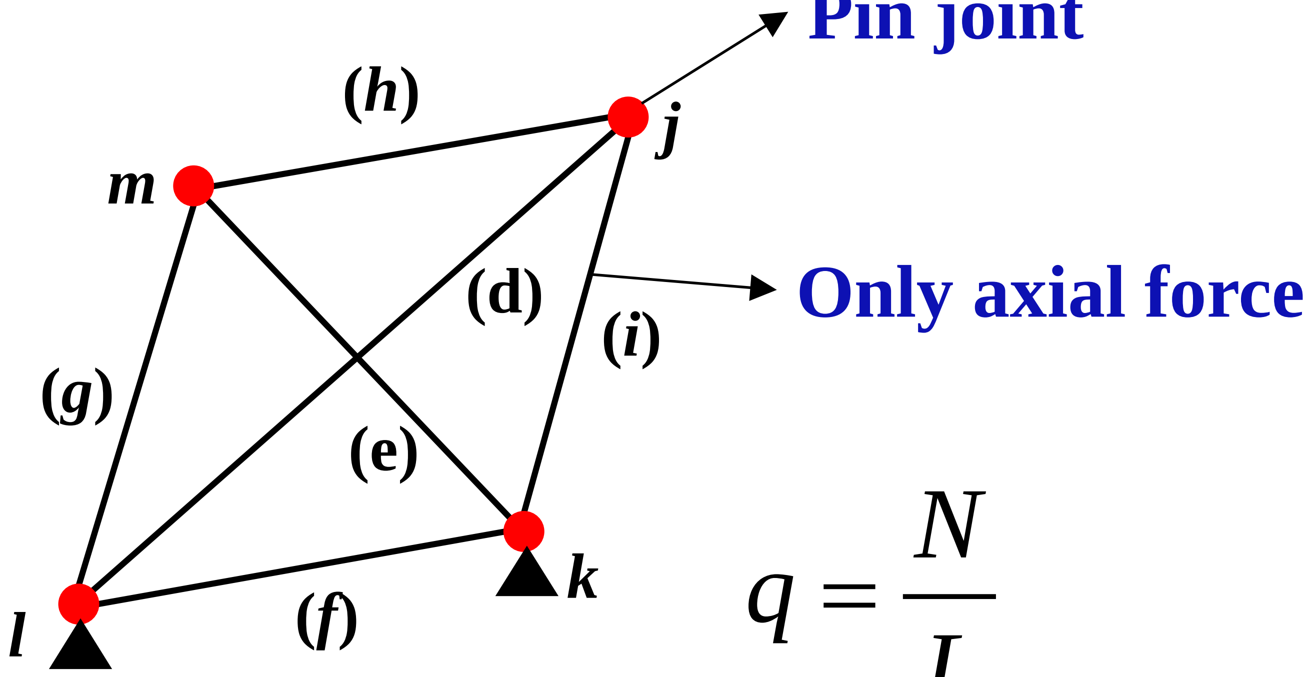

**Minimize :** $\boldsymbol{U}^T \left( \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{d} \right) \boldsymbol{K} \left( \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{d} \right) \boldsymbol{U} \left( \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{d} \right)$

**Subject to:** $V \left( \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{d} \right) \leq V_{upper} \left( \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{d} \right), d_{lower} \leq d \leq d_{upper},$

$\boldsymbol{X}_{lower} \leq \boldsymbol{X} \leq \boldsymbol{X}_{upper}, \boldsymbol{Y}_{lower} \leq \boldsymbol{Y} \leq \boldsymbol{Y}_{upper}$

# *Introducing FDM*

## Truss structure




$$q = \frac{N}{L}$$

**Shape optimization:**
**optimal nodal location**

↓

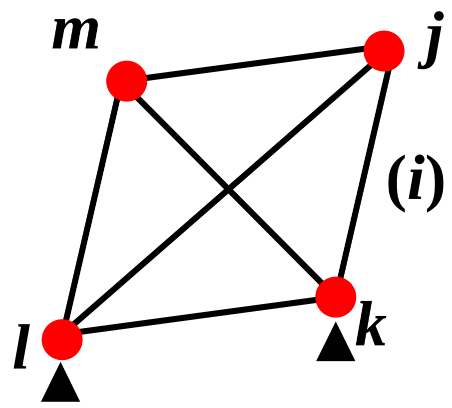
**Only force equilibrium**

↓

**Determined by *q***

# *Introducing FDM*

**Denote the connectivity matrix as C , where**




$$C_{(i,p)} = \begin{cases} 1 & p = j \\ -1 & p = k \\ 0 & \text{other case} \end{cases}$$

***i*=1,2, …, m; *j*, *k*=1, 2, …, n**

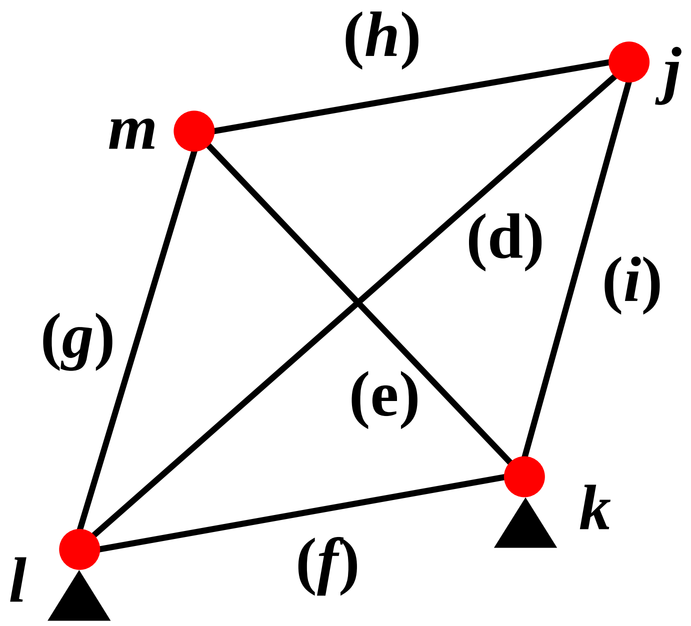
**Divide nodes to free and fix nodes**
**And rearrange the matrix C**

$$C = \begin{bmatrix} \overbrace{C_{1,1} & \cdots & C_{1,nfree}}^{\mathbf{C_{free}}} & \overbrace{\cdots & C_{1,n}}^{\mathbf{C_{fix}}} \\ \vdots & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ C_{m,1} & \cdots & \cdots & \cdots & C_{m,n} \end{bmatrix} = \begin{bmatrix} C_{\text{free}} & C_{\text{fix}} \end{bmatrix}$$

# *Introducing FDM*

## Denote the force density matrix as Q , where

$$\boldsymbol{Q} = (\boldsymbol{C}_{\text{free}}, \boldsymbol{C}_{\text{fix}})^T \operatorname{diag}(\boldsymbol{q})(\boldsymbol{C}_{\text{free}}, \boldsymbol{C}_{\text{fix}}) = \begin{bmatrix} \boldsymbol{C}_{\text{free}}^T \operatorname{diag}(\boldsymbol{q}) C_{\text{free}} & \boldsymbol{C}_{\text{free}}^T \operatorname{diag}(\boldsymbol{q}) C_{\text{fix}} \\ \boldsymbol{C}_{\text{fix}}^T \operatorname{diag}(\boldsymbol{q}) C_{\text{free}} & \boldsymbol{C}_{\text{fix}}^T \operatorname{diag}(\boldsymbol{q}) C_{\text{fix}} \end{bmatrix}$$




**Force density vector**

$\boldsymbol{q} = (q_d, q_e, q_f, q_g, q_h, q_i)$

# *Nodal location*

## Determination of nodal location

**By introducing FDM, the location of free nodes can be derived from the following equation**

$$\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{free}} \boldsymbol{x}_{\text{free}} = -\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{fix}} \boldsymbol{x}_{\text{fix}} + \boldsymbol{P}_{x,\text{free}}$$

$$\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{free}} \boldsymbol{y}_{\text{free}} = -\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{fix}} \boldsymbol{y}_{\text{fix}} + \boldsymbol{P}_{y,\text{free}}$$

**No external load on free nodes**

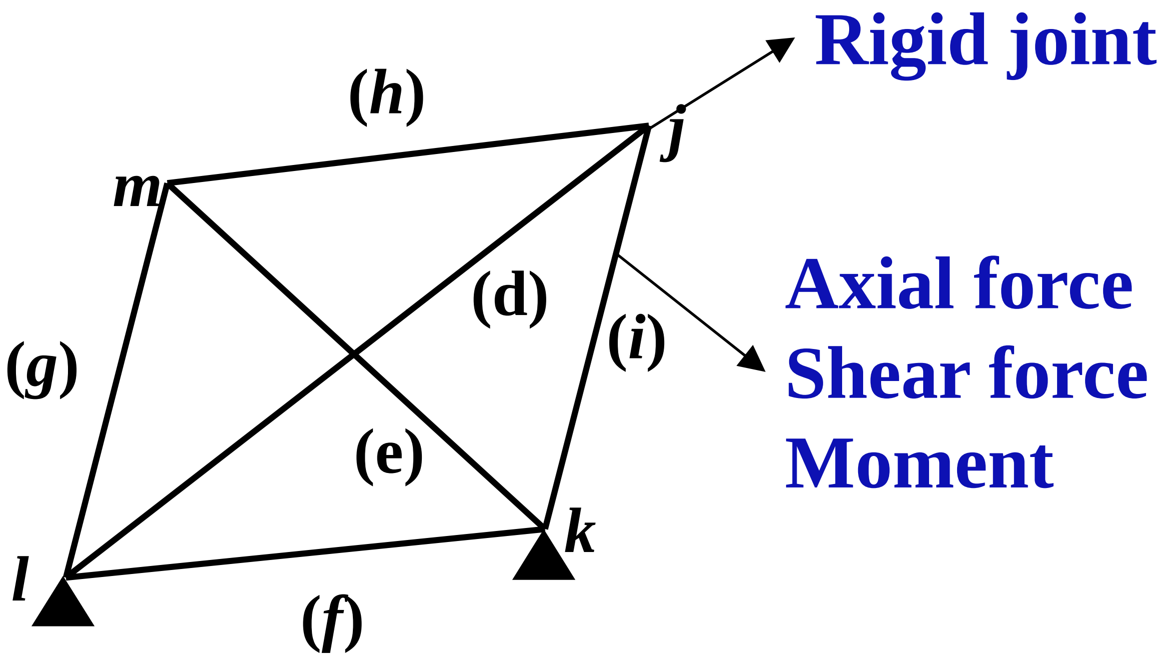
**Functions of *q***

$$\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{free}} \boldsymbol{x}_{\text{free}} = -\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{fix}} \boldsymbol{x}_{\text{fix}}$$

$$\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{free}} \boldsymbol{y}_{\text{free}} = -\boldsymbol{C}_{\text{free}}^T \text{diag}(\boldsymbol{q}) \boldsymbol{C}_{\text{fix}} \boldsymbol{y}_{\text{fix}}$$

# *Problem Reformulation*

**Frame structure**




**Same discretization with Truss**

**FDM (*q*)**

**Optimal shape of Frame**

***Minimize :*** $U^T\left(X(q),Y(q),d\right)K\left(X(q),Y(q),d\right)U\left(X(q),Y(q),d\right)$

***Subject to:*** $V\left(X(q),Y(q),d\right)\le V_{upper},\ \mathbf{d}_{lower}\le\mathbf{d}\le\mathbf{d}_{upper},\ \mathbf{q}_{lower}\le\mathbf{q}\le\mathbf{q}_{upper}$

# *Sensitivity Analysis*

## For objective function

$$\frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial X_j} \times \frac{\partial X_j}{\partial q_i} + \frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial Y_j} \times \frac{\partial Y_j}{\partial q_i}; \qquad \frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial d_i} = \boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial d_i} \boldsymbol{U}$$

$$\frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial X_j} = \boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial X_j} \boldsymbol{U}; \quad \frac{\partial \boldsymbol{U}^T \boldsymbol{K}\boldsymbol{U}}{\partial Y_j} = \boldsymbol{U}^T \frac{\partial \boldsymbol{K}}{\partial Y_j} \boldsymbol{U};$$

**Adjoint variable method**

$\frac{\partial X_j}{\partial q_i}, \frac{\partial Y_j}{\partial q_i}$ : **Only free nodes are considered**

# *Sensitivity Analysis*

$$\frac{\partial X_{free}}{\partial q_i} = -\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{free}\right)^{-1}\left(\frac{\partial\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{free}\right)}{\partial q_i}X_{free} + \frac{\partial\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{fix}\right)}{\partial q_i}X_{fix}\right)$$

$$\frac{\partial Y_{free}}{\partial q_i} = -\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{free}\right)^{-1}\left(\frac{\partial\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{free}\right)}{\partial q_i}Y_{free} + \frac{\partial\left(\boldsymbol{C}^T_{free}\boldsymbol{QC}_{fix}\right)}{\partial q_i}Y_{fix}\right)$$

## For Volume constraint

$$\frac{\partial V\left(X\left(\boldsymbol{q}\right),Y\left(\boldsymbol{q}\right),\boldsymbol{d}\right)}{\partial q_i} = \sum_{i=1}^{n} A_i\left(\sum_{i_k=1}^{s}\frac{\partial L_i}{\partial X_{i_k}}\bullet\frac{\partial X_{i_k}}{\partial q_i} + \frac{\partial L_i}{\partial Y_{i_k}}\bullet\frac{\partial Y_{i_k}}{\partial q_i}\right); \quad \frac{\partial V\left(X\left(\boldsymbol{q}\right),Y\left(\boldsymbol{q}\right),\boldsymbol{d}\right)}{\partial d_i} = \frac{\partial A_i}{\partial di}L_i$$

# *Further Improvement*

***Minimize :*** $\boldsymbol{U}^{T}\left(\boldsymbol{X}(\boldsymbol{q}),\boldsymbol{Y}(\boldsymbol{q}),\boldsymbol{d}\right)\boldsymbol{K}\left(\boldsymbol{X}(\boldsymbol{q}),\boldsymbol{Y}(\boldsymbol{q}),\boldsymbol{d}\right)\boldsymbol{U}\left(\boldsymbol{X}(\boldsymbol{q}),\boldsymbol{Y}(\boldsymbol{q}),\boldsymbol{d}\right)$

***Subject to:*** $V\left(\boldsymbol{X}(\boldsymbol{q}),\boldsymbol{Y}(\boldsymbol{q}),\boldsymbol{d}\right)\leq V_{upper},\boldsymbol{d}_{lower}\leq\boldsymbol{d}\leq\boldsymbol{d}_{upper},\boldsymbol{q}_{lower}\leq\boldsymbol{q}\leq\boldsymbol{q}_{upper}$

**Optimal result with thin element and closely spaced nodes**

**Delete** **Merge**

***Minimize :*** $\boldsymbol{U}^{T}\left(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{d}\right)\boldsymbol{K}\left(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{d}\right)\boldsymbol{U}\left(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{d}\right)$

***Subject to:*** $V\left(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{d}\right)\leq V_{upper},\boldsymbol{d}_{lower}\leq\boldsymbol{d}\leq\boldsymbol{d}_{upper},$

$\boldsymbol{X}_{lower}\leq\boldsymbol{X}\leq\boldsymbol{X}_{upper},\boldsymbol{Y}_{lower}\leq\boldsymbol{Y}\leq\boldsymbol{Y}_{upper}$

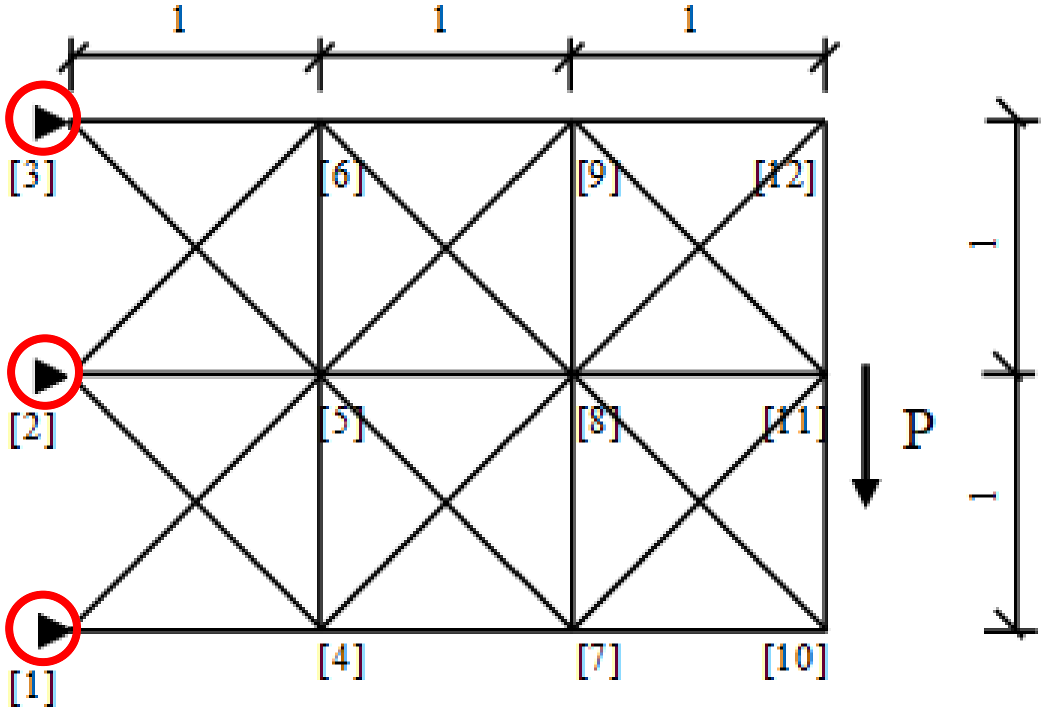
# ***Flow chart***

Initial information of structure → Set initial value for force density and element diameter → Calculate the sensitivities → Obtain the converge result

→ Delete the thin element and merge the closely spaced nodes for further improvement → Obtain the distinct structural layout of the optimal solution

# *Numerical Example* 1

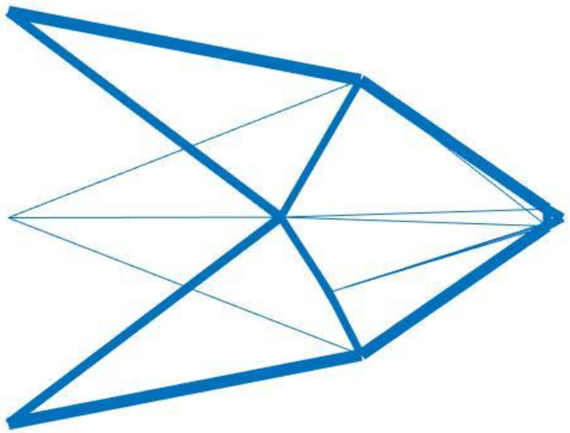
- **Cantilever beam**

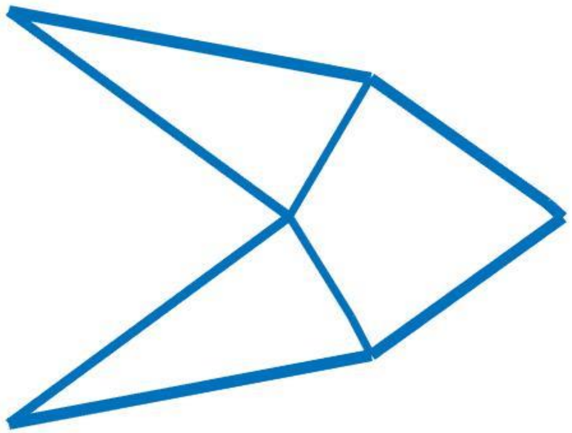


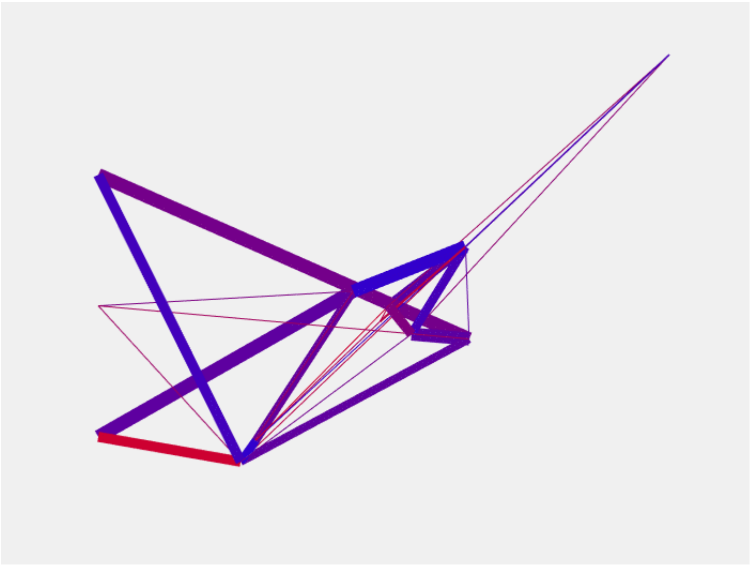

**Pin support**

$d_{lower}$ = 0.001 and $d_{upper} = \infty$
$q_{lower}$ = -1000 and $q_{upper}$ = 1000
$V_{upper}$ = 1

# *Numerical Example* 1



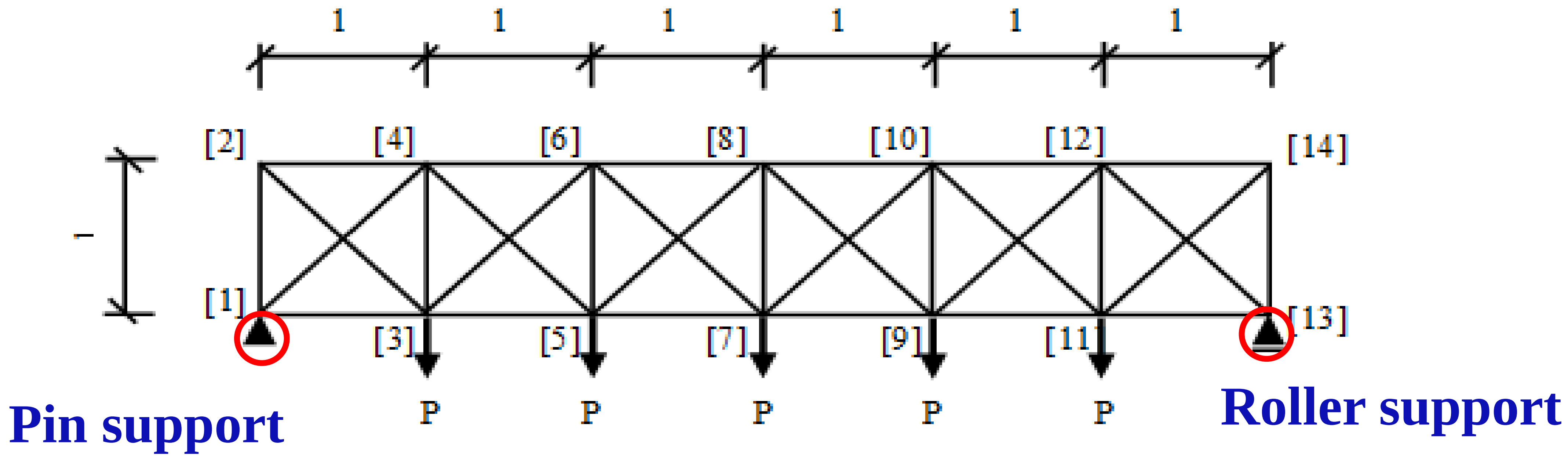




| Solution | Volume | Compliance |
|---|---|---|
| Before | 1 | 83.1183 |
| After | 1 | 82.0866 |

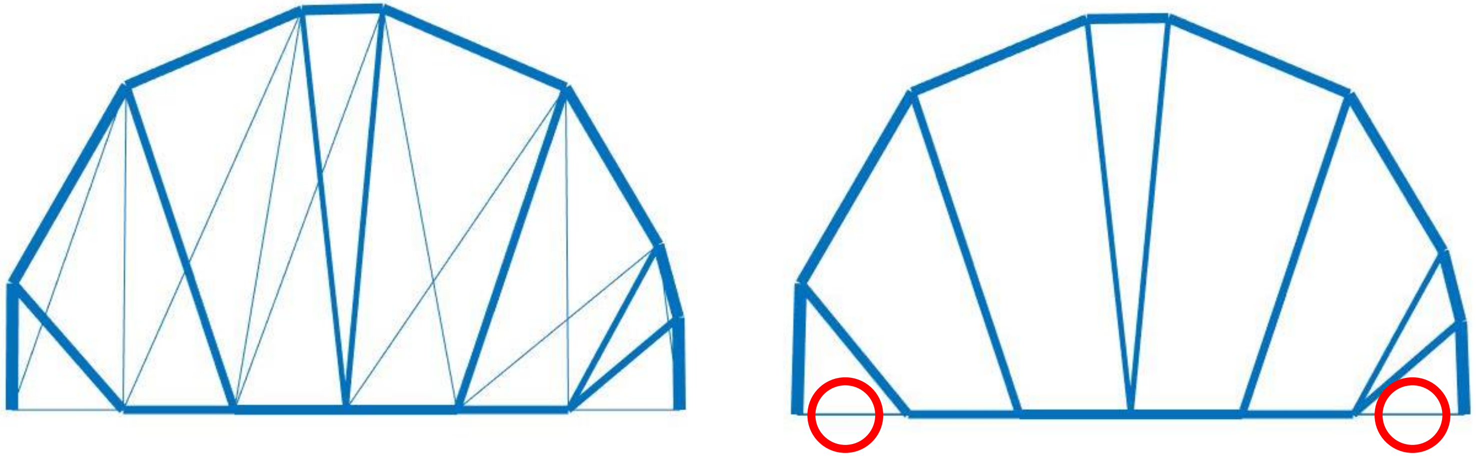
# *Numerical Example 2*

- **Bridge beam**

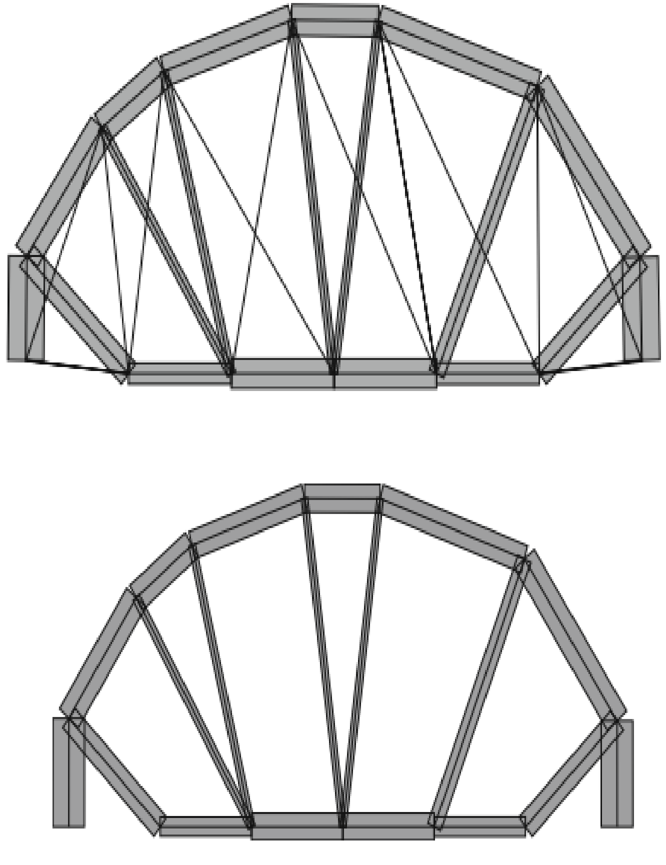



$d_{lower}$ = 0.001 and $d_{upper} = \infty$; $q_{lower}$ = -1000 and $q_{upper}$ = 1000; $V_{upper}$ = 1

# *Numerical Example 2*

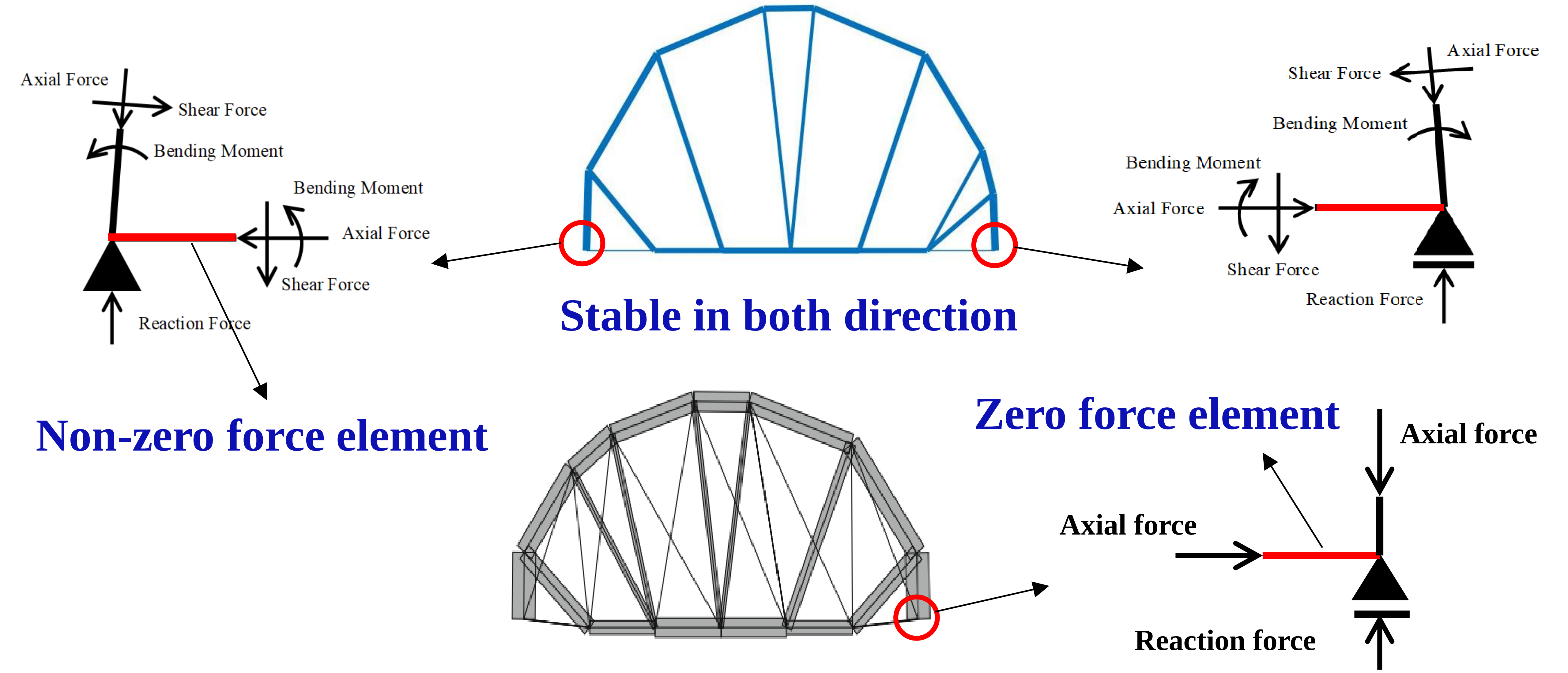


| Solution | Volume | Compliance |
| --- | --- | --- |
| Before | 1 | 1221.0307 |
| After | 1 | 1219.2239 |



**Source**: Ohsaki M, Hayashi K. Force density method for simultaneous optimization of geometry and topology of trusses. Struct Multidiscip Optim, 2017

# *Numerical Example 2*



# *Conclusion*

- **Brief Summary**

**The proposed method includes：**

- **Shape and topology optimization**
- **Force density method**
- **Further optimization (Filter)**

**And has the following conclusion:**

- **Melting nodes can be avoid by restricting *q***
- **Stable optimal result can be obtained by using beam element**

# ***Thanks for your kind attention***

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**Date : 2019.05.21**
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