WCSMO-14

14th World Congress of Structural and Multidisciplinary Optimization

Worst-case design of plane frames using order statistics

Wei Shen[1], Makoto Ohsaki[1], Makoto Yamakawa[2]

1. **Department of Architecture and Architectural Engineering, Kyoto University**
2. **Department of Architecture, Tokyo University of Science**

Presenter: Wei Shen

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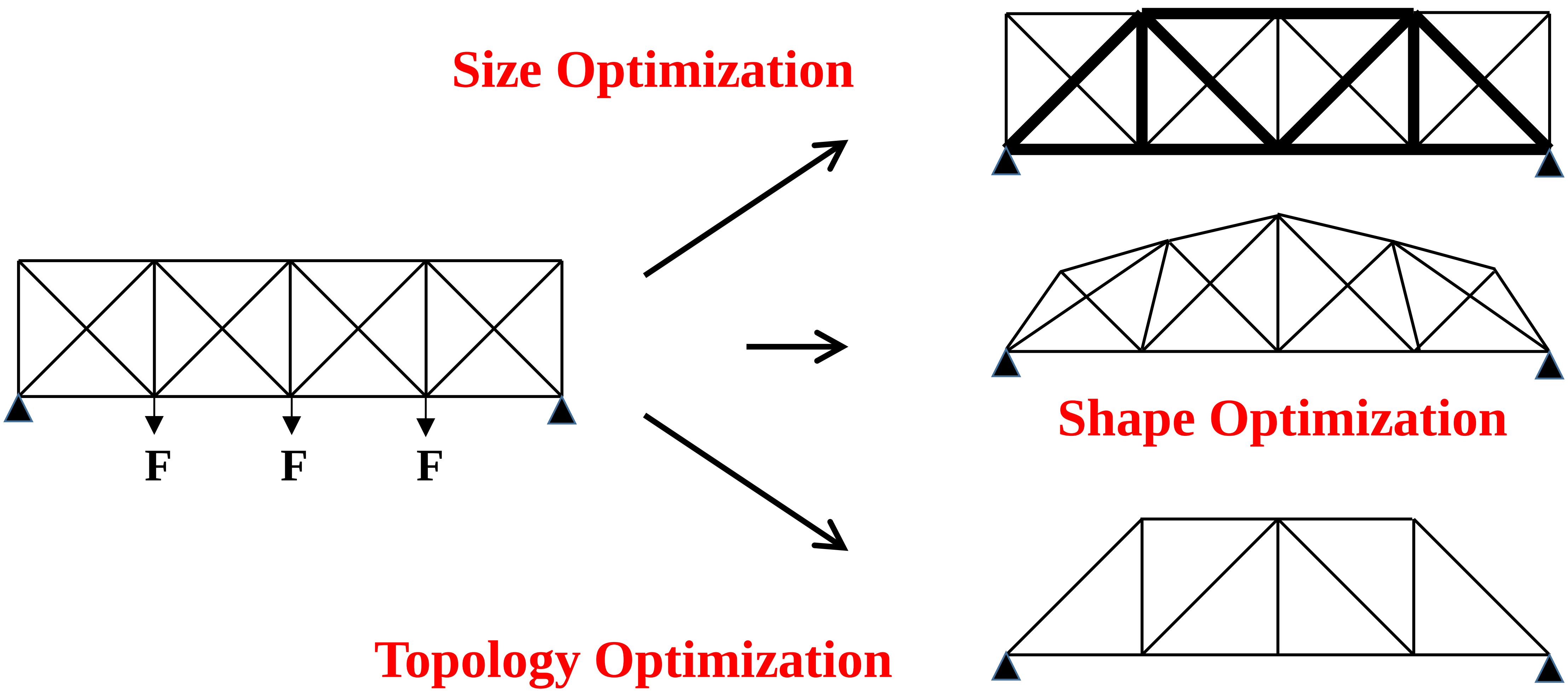
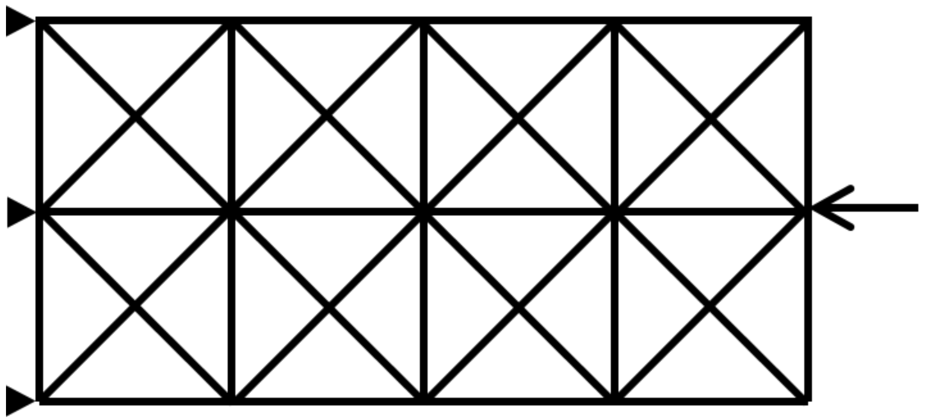
Background



Size Optimization
F
F
F
Shape Optimization
Topology Optimization

Background

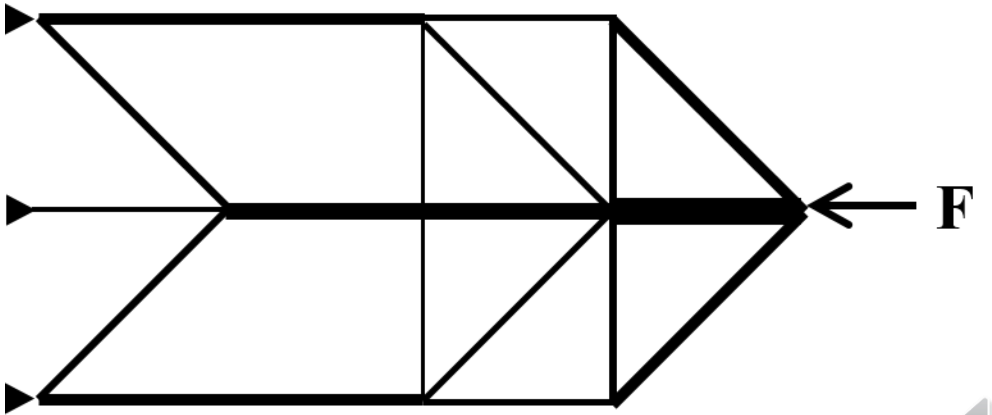
Uncertainty in real world



Suppose uncertainty exists in Young’s modulus

Deterministic design

Considering Uncertainty



Source: A. Asadpoure *et al*, Robust topology optimization of structures with uncertainties in stiffness - Application to truss structures, 2011, Computers and Structures.

Existing Challenges

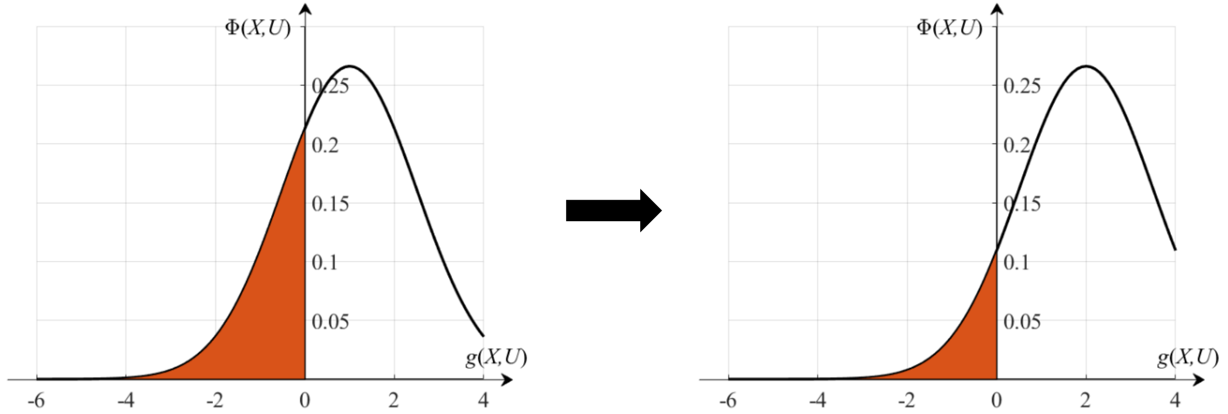
(1) Unknown distribution of uncertainty

Two kinds of probability-based design

(i) Reliability-based design

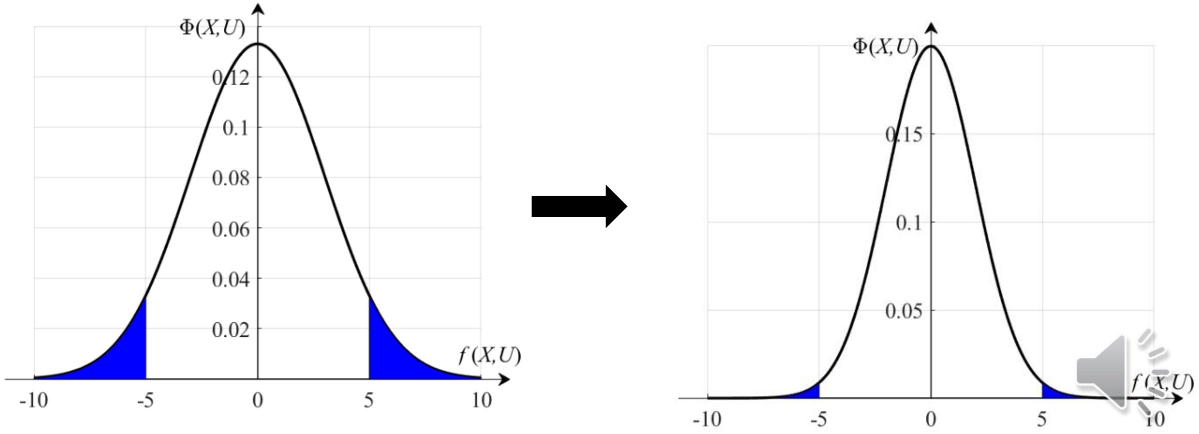
Minimize: $f\left(\boldsymbol{X};\boldsymbol{U}\right)$

Subject to: $P_{\text{fail}} = \int_{g(\boldsymbol{X};\boldsymbol{U})<0} \Phi\left(\boldsymbol{X};\boldsymbol{U}\right) d\boldsymbol{U} \leq \bar{P}$



(ii) Robust design

Minimize: $E\left(f\left(\boldsymbol{X};\boldsymbol{U}\right)\right)+k\sigma\left(f\left(\boldsymbol{X};\boldsymbol{U}\right)\right)$

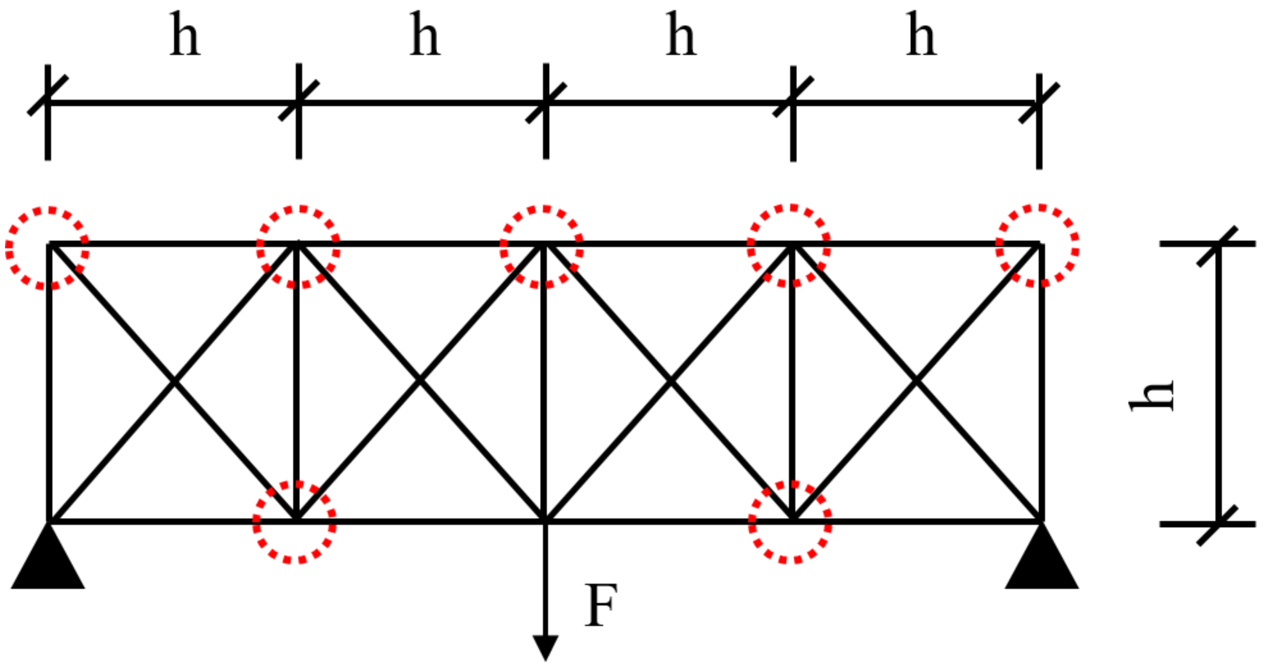


Existing Challenges

(2) Exact value of worst-case event

Handle the uncertainty with worst-case design

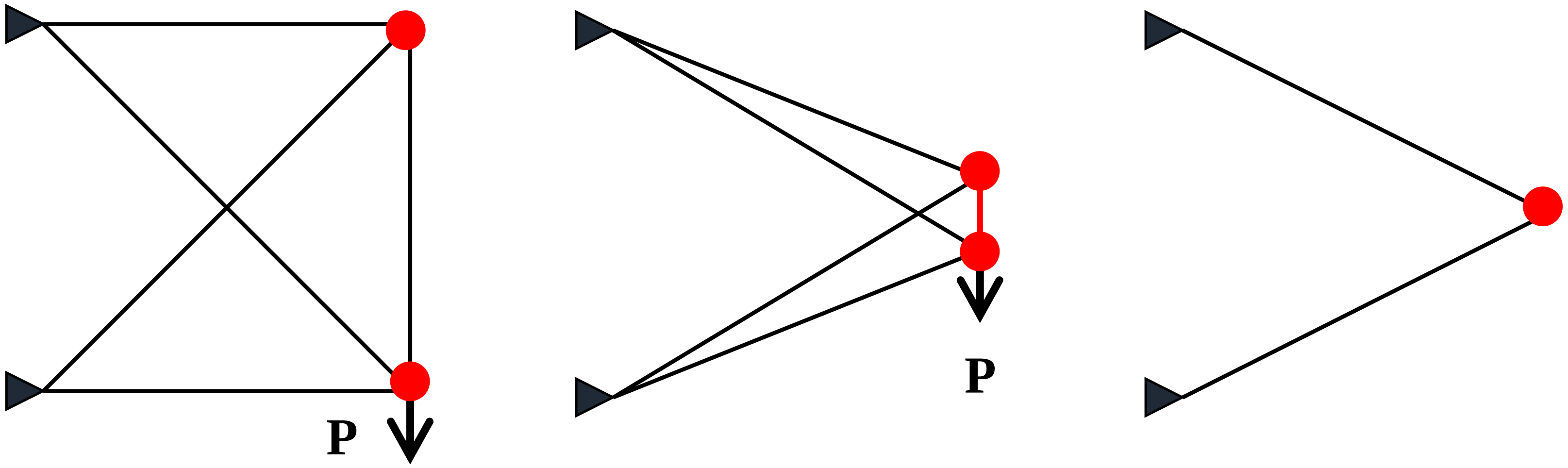
$$\text{Minimize: } f^{\max}(\boldsymbol{X};\boldsymbol{U}) = \max_{\boldsymbol{U}\in\Theta} f(\boldsymbol{X};\boldsymbol{U})$$



Hard to obtain the exact worst structural response even if Θ is simple

Existing Challenges

(3) Melting nodes



Stiffness Matrix

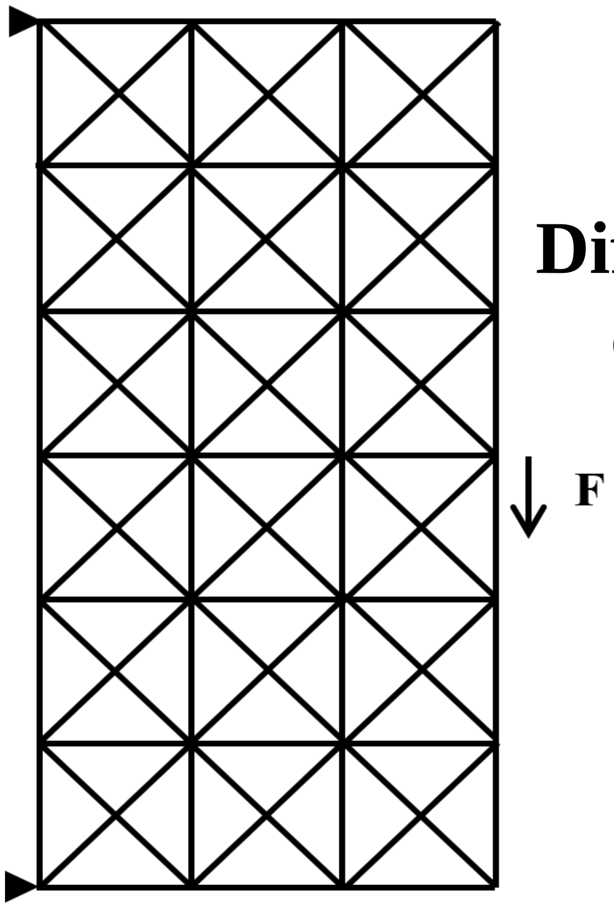
Red nodes become close

Singular

Desirable to avoid existence of extremely short member

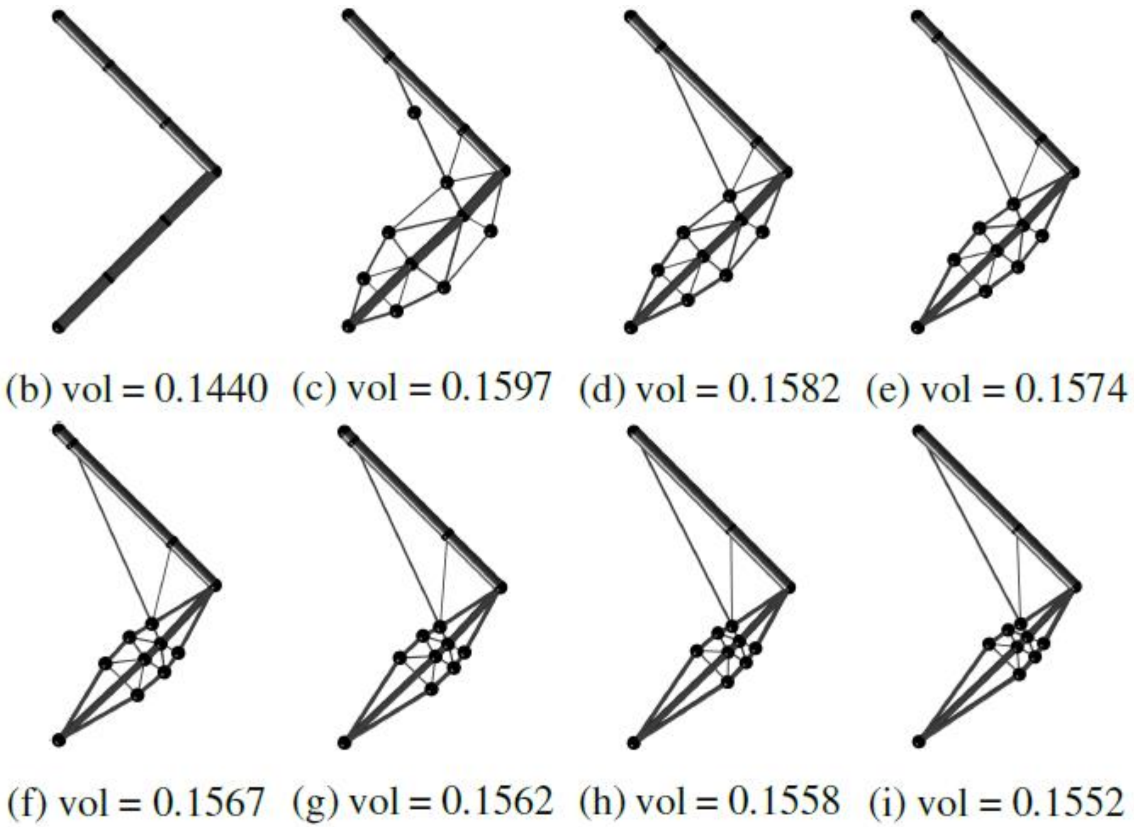
Existing Challenge

(4) Stability constraint



Different global constraints

Optimal solution with slender elements



Source: AG. Weldeyesus *et al*, Truss geometry and topology optimization with global stability constraints, 2020, Structural and Multidisciplinary Optimization

Optimization with global stability

$$\text{Minimize } \sigma = \max_{\substack{i=1,2,\cdots m_e \\ j=1,2,\cdots p}} \left(\sigma_{V,ij}\left(\boxed{\boldsymbol{x}_{\text{free}}, \boldsymbol{y}_{\text{free}}}, \boldsymbol{A}\right)\right)$$

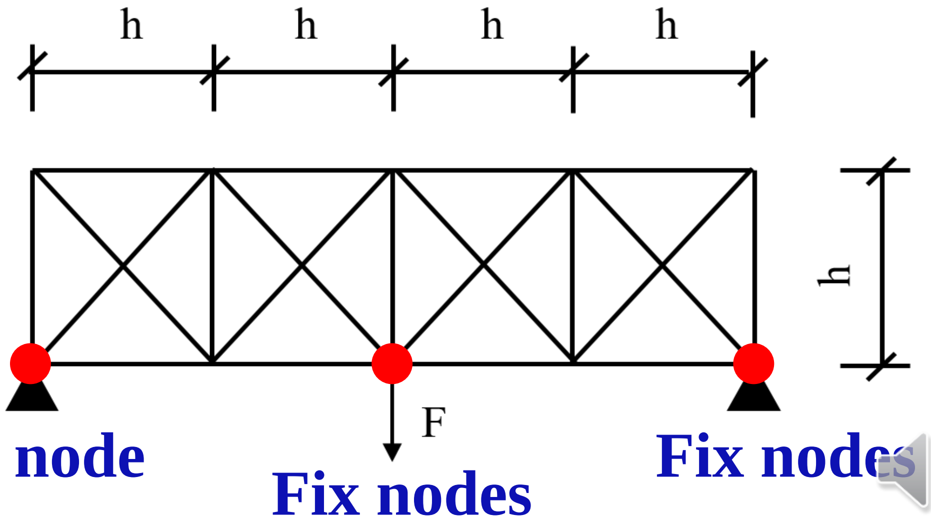
$$\text{subject to } \frac{1}{\boxed{\lambda^{\text{cr}}}\left(\boldsymbol{x}_{\text{free}}, \boldsymbol{y}_{\text{free}}, \boldsymbol{A}\right)} \le \frac{1}{\lambda_{\text{L}}}; \quad V\left(\boldsymbol{x}_{\text{free}}, \boldsymbol{y}_{\text{free}}, \boldsymbol{A}\right) \le V_{\text{U}};$$

$$\underline{\boldsymbol{x}}_{\text{free}} \le \boldsymbol{x}_{\text{free}} \le \overline{\boldsymbol{x}}_{\text{free}}; \quad \underline{\boldsymbol{y}}_{\text{free}} \le \boldsymbol{y}_{\text{free}} \le \overline{\boldsymbol{y}}_{\text{free}}; \quad \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \overline{\boldsymbol{A}}$$

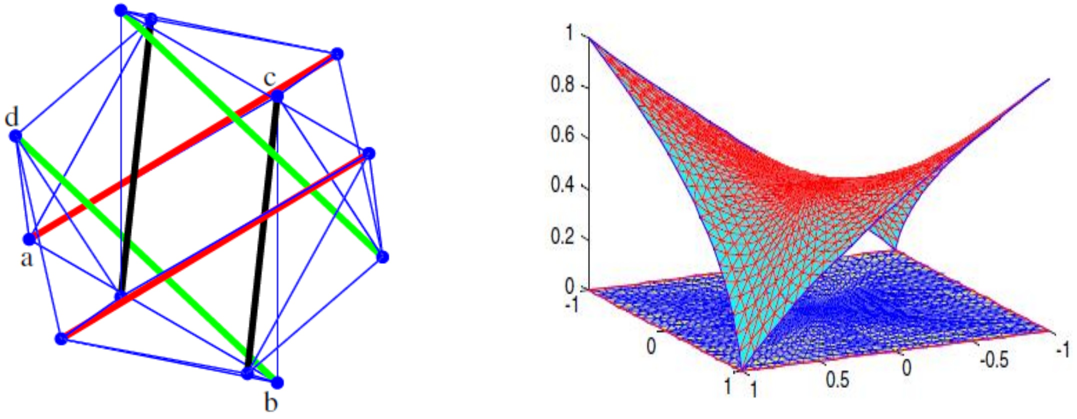
$\boldsymbol{x}_{\text{free}}$: x **coordinates of free nodes;**
$\boldsymbol{y}_{\text{free}}$: y **coordinates of free nodes;**
$\boldsymbol{A}$ **: Cross-sectional areas**
$\boldsymbol{V}$ **: Structural volume**
$\boldsymbol{K}$ **: Elastic stiffness matrix**
$\boldsymbol{K}_{\text{G}}$ **: Geometry stiffness matrix**

Smallest positive eigenvalue:

$$\left(\boldsymbol{K}\left(\boldsymbol{x}_{\text{free}}, \boldsymbol{y}_{\text{free}}, \boldsymbol{A}\right) - \lambda \boldsymbol{K}_G\left(\boldsymbol{x}_{\text{free}}, \boldsymbol{y}_{\text{free}}, \boldsymbol{A}\right)\right)\boldsymbol{\Phi} = 0$$



Force density method for shape optimization



Source: (left) JY Zhang, M Ohsaki. Adaptive force density method for form-finding problem of tensegrity structures. (Right) MO Ruy *et* al The natural force density method for the shape finding of taut structures

FDM is widely used in form-finding of tension and tensegrity structure

$$q = \frac{N}{L}$$

Axial force

Force density

Member length

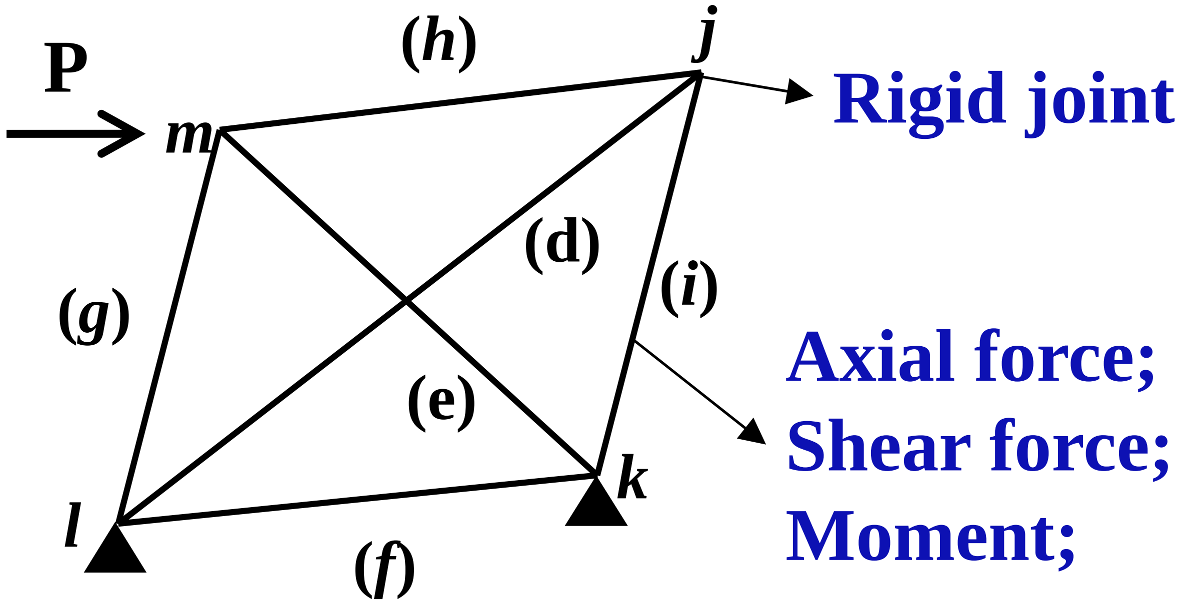
Diagonal matrix

$$\boldsymbol{x}_{\text{free}} = -\left(\tilde{\boldsymbol{Q}}_{\text{free}}^T \boxed{\operatorname{diag}(\boldsymbol{q})} \tilde{\boldsymbol{Q}}_{\text{free}}\right)^{-1} \tilde{\boldsymbol{Q}}_{\text{free}}^T \operatorname{diag}(\boldsymbol{q}) \tilde{\boldsymbol{Q}}_{\text{fix}} \boldsymbol{x}_{\text{fix}}$$

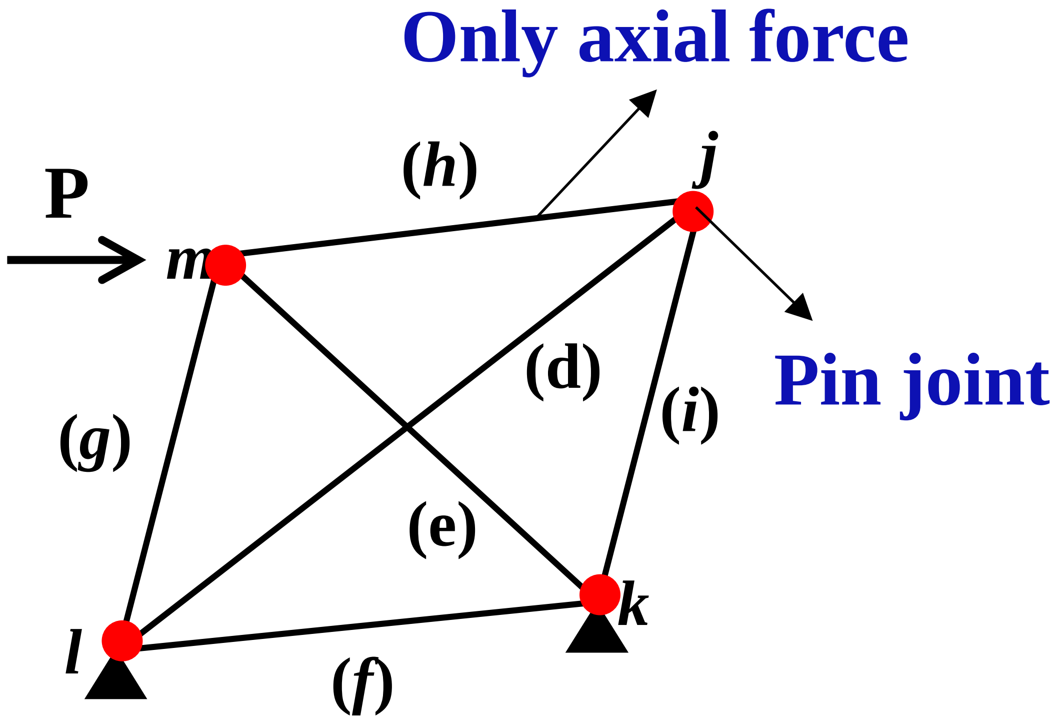
$$\boldsymbol{y}_{\text{free}} = -\left(\tilde{\boldsymbol{Q}}_{\text{free}}^T \operatorname{diag}(\boldsymbol{q}) \tilde{\boldsymbol{Q}}_{\text{free}}\right)^{-1} \boxed{\tilde{\boldsymbol{Q}}_{\text{free}}^T} \operatorname{diag}(\boldsymbol{q}) \boxed{\tilde{\boldsymbol{Q}}_{\text{fix}}} \boldsymbol{y}_{\text{fix}}$$

Connectivity matrix

Introducing Force density method



Frame structure



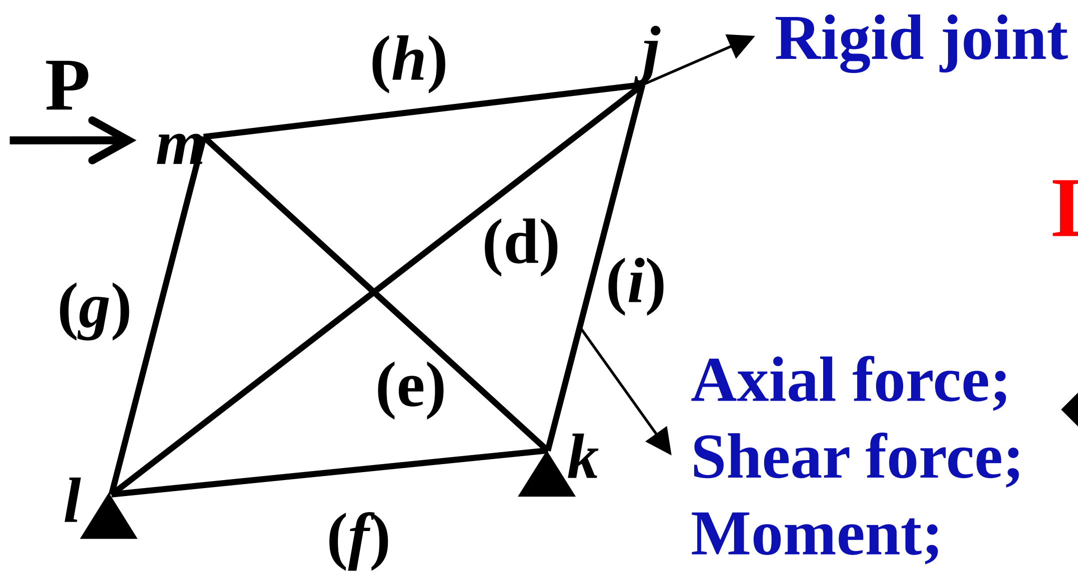
Auxiliary truss structure

FDM can not be directly applied

Used to determine the location of nodes of frame structure

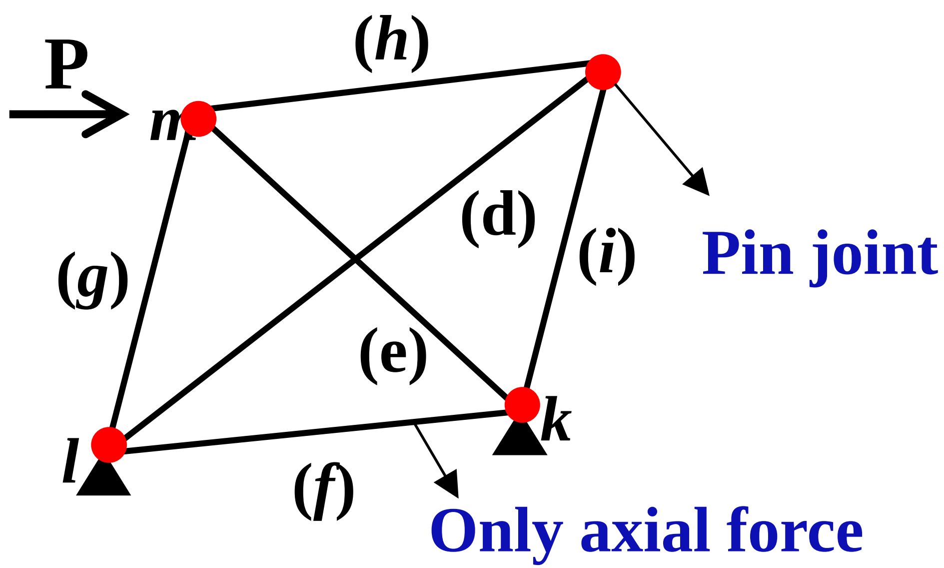
Introducing Force density method

Frame structure



Location of nodes

Auxiliary truss structure



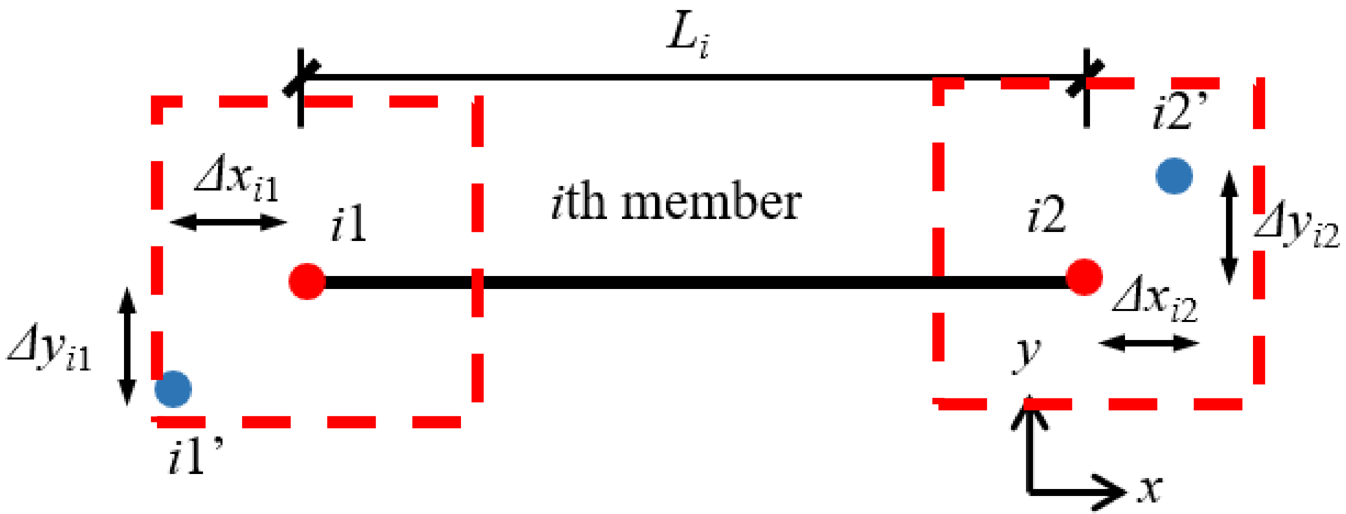
Minimize : $\sigma = \max\limits_{\substack{i=1,2,\cdots m_e \\ j=1,2,\cdots p}} \left(\sigma_{V,ij}\left(\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{A}\right)\right)$

$\boxed{\gamma^{\mathrm{cr}} = 1/\lambda^{\mathrm{cr}}}$ $\boxed{\gamma_{\mathrm{U}} = 1/\lambda_{\mathrm{L}}}$

Subject to: $\gamma^{\mathrm{cr}}\left(\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{A}\right) \le \gamma_{\mathrm{U}};\; V\left(\boldsymbol{x}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{y}_{\mathrm{free}}(\boldsymbol{q}), \boldsymbol{A}\right) \le V_{\mathrm{U}}; \underline{\boldsymbol{q}} \le \boldsymbol{q} \le \overline{\boldsymbol{q}};\; \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \overline{\boldsymbol{A}}$

Uncertainty in member stiffness

Uncertainty in nodal locations



Nominal locations

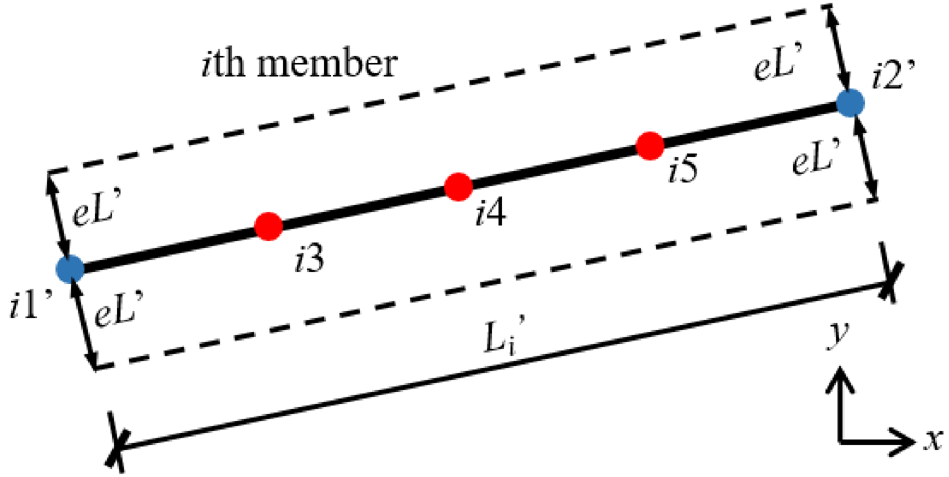
$$\left(x_j', y_j'\right) = \left(x_j, y_j\right) + \left(\Delta x_j, \Delta y_j\right)$$

Randomness

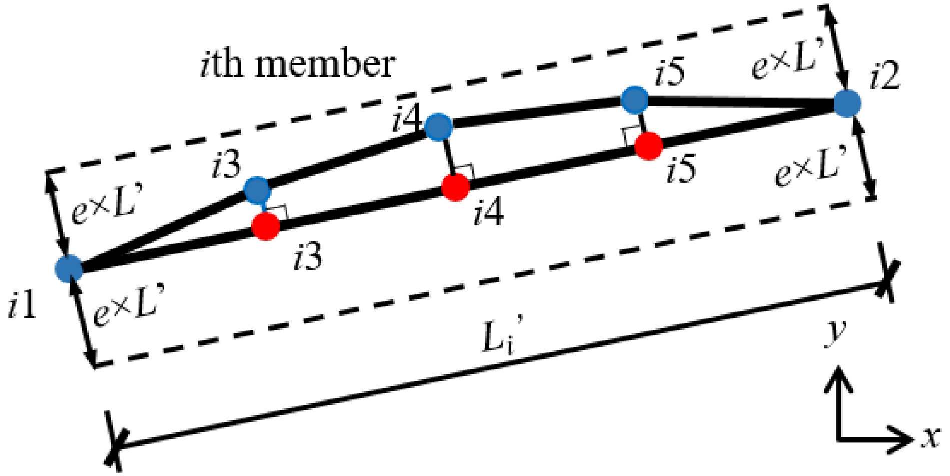
Uncertain locations

Uncertainty in member stiffness

Uncertainty in intermediate nodal locations



Uncertainty

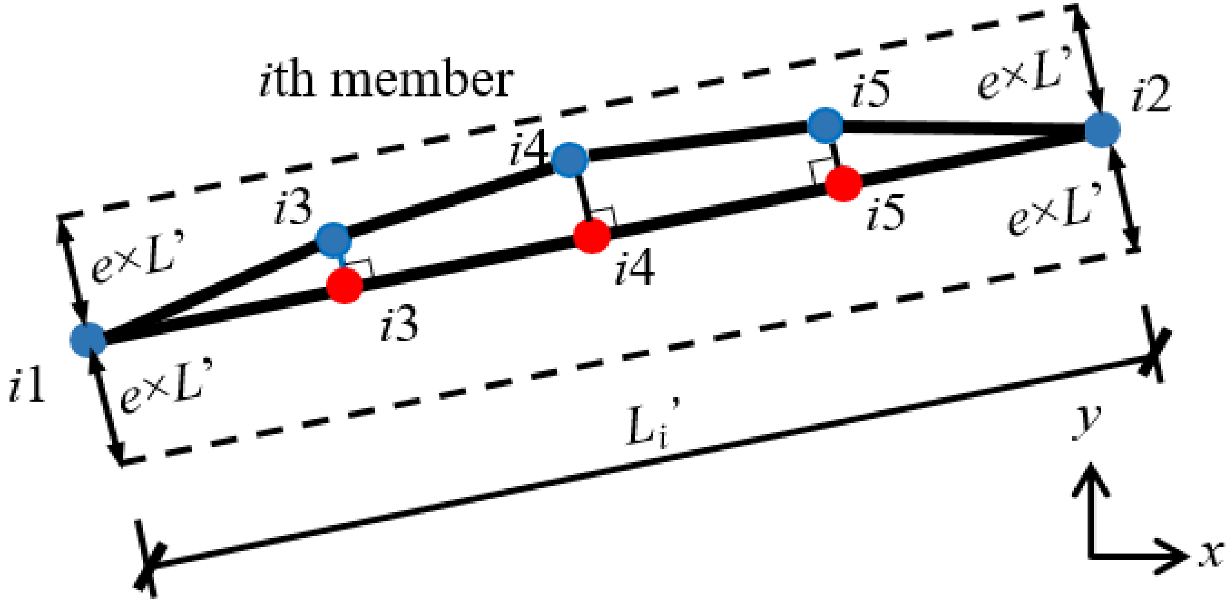


Four beam elements for one member

Vertical within eccentricity *e*

Uncertainty in member stiffness

Uncertainty in cross-sectional areas



$$A_i' = A_i + \Delta A_i$$

A' for all elements in member *i*

Approximate worst-case using order statistics

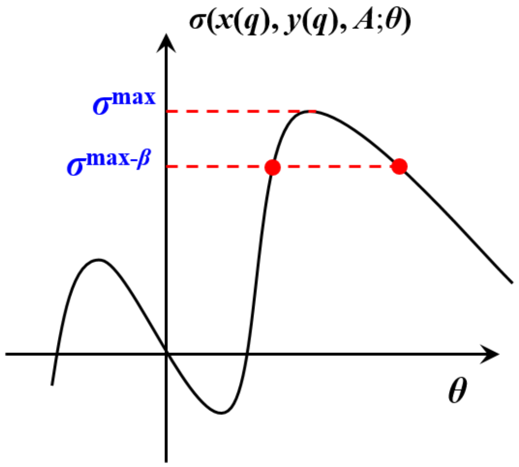
Worst-case design with uncertainties $\boldsymbol{\theta}=\left(\Delta\boldsymbol{x},\Delta\boldsymbol{y},\Delta\boldsymbol{A}\right)$

$$\text{Minimize } \sigma^{\max}=\boxed{\max_{\boldsymbol{\theta}\in\Omega}\sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}),\boldsymbol{y}_{\text{free}}(\boldsymbol{q}),\boldsymbol{A};\boldsymbol{\theta}\right)}$$

$$\text{subject to } \gamma^{\text{cr,max}}=\boxed{\max_{\boldsymbol{\theta}\in\Omega}\left(\gamma^{\text{cr}}\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}),\boldsymbol{y}_{\text{free}}(\boldsymbol{q}),\boldsymbol{A};\boldsymbol{\theta}\right)\right)}\le\gamma_{\text{U}};\quad V\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}),\boldsymbol{y}_{\text{free}}(\boldsymbol{q}),\boldsymbol{A}\right)\le V_{\text{U}};$$

$$\underline{\boldsymbol{q}}\le\boldsymbol{q}\le\bar{\boldsymbol{q}};\quad\underline{\boldsymbol{A}}\le\boldsymbol{A}\le\bar{\boldsymbol{A}}$$

Relaxed



100βth ($0<\beta<1$) quantile of stress $\sigma^{\text{max-}\beta}$ and $\gamma^{\text{max-}\beta}$

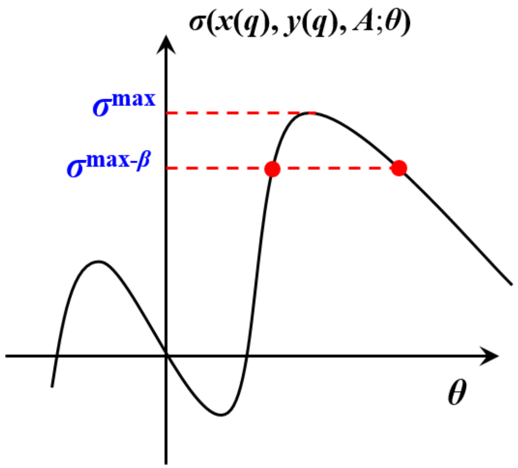
$$\text{Probability}\left\{\sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}),\boldsymbol{y}_{\text{free}}(\boldsymbol{q}),\boldsymbol{A};\boldsymbol{\theta}\right)\le\boxed{\sigma^{\max-\beta}}\right\}=\beta$$

$$\text{Probability}\left\{\gamma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}),\boldsymbol{y}_{\text{free}}(\boldsymbol{q}),\boldsymbol{A};\boldsymbol{\theta}\right)\le\boxed{\gamma^{\max-\beta}}\right\}=\beta$$

Approximate worst-case using order statistics

$$\text{Probability}\left\{\sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}\right) \le \sigma^{\max-\beta}\right\} = \beta$$

$$\text{Probability}\left\{\gamma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}\right) \le \gamma^{\max-\beta}\right\} = \beta$$



Given *m* sets of θ_1, θ_2, …, θ_m , obtain *m* response

$$\sigma_1 = \sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}_1\right), \cdots, \sigma_m = \sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}_m\right)$$

$$\gamma_1 = \sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}_1\right), \cdots, \gamma_m = \sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A};\boldsymbol{\theta}_m\right)$$

and place them in a descending order

$$\sigma_{1:m} \ge \ldots \ge \sigma_{k:m} \ge \ldots \ge \sigma_{m:m}; \gamma_{1:m} \ge \ldots \ge \gamma_{k:m} \ge \ldots \ge \gamma_{m:m}; 1 \le k \le m$$

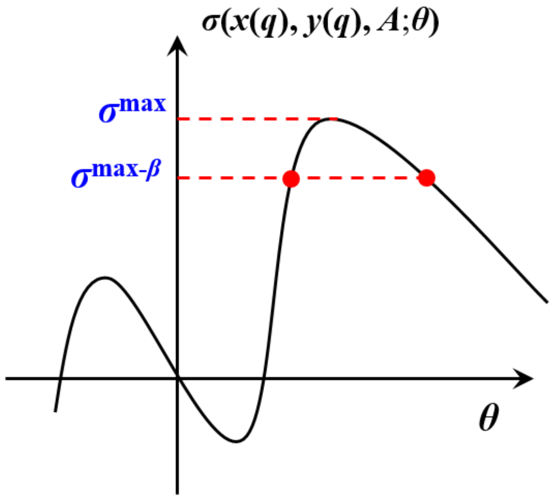
Approximate worst-case using order statistics

Based on the statistical inference theory of order statistics

$$\alpha_k = \Pr\left\{F_\sigma\left(\sigma_{k:m}^{\mathrm{cr}}\right) \geq \beta\right\} = \sum_{r=0}^{m-k} \binom{m}{r} \beta^r \left(1-\beta\right)^{m-r}$$

$$\alpha_k = \Pr\left\{F_\gamma\left(\gamma_{k:m}^{\mathrm{cr}}\right) \geq \beta\right\} = \sum_{r=0}^{m-k} \binom{m}{r} \beta^r \left(1-\beta\right)^{m-r}$$

Relation between *k* and *β*



Relation between k and β ($\alpha_k = 0.9$, $m = 200$)

k	1	2	3	4	5	6	7	8	9	10
β	0.989	0.981	0.974	0.967	0.960	0.954	0.948	0.942	0.936	0.930
k	11	12	13	14	15	16	17	18	19	20
β	0.924	0.918	0.912	0.907	0.901	0.895	0.890	0.884	0.878	0.873

Approximate worst-case using order statistics

$$\text{Minimize } \sigma^{\max} = \max_{\theta\in\Omega} \sigma\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}; \boldsymbol{\theta}\right)$$

$$\text{subject to } \gamma^{\text{cr,max}} = \max_{\theta\in\Omega}\left(\gamma^{\text{cr}}\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}; \boldsymbol{\theta}\right)\right) \le \gamma_{\text{U}};\quad V\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}\right) \le V_{\text{U}};$$

$$\underline{\boldsymbol{q}} \le \boldsymbol{q} \le \bar{\boldsymbol{q}};\quad \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \bar{\boldsymbol{A}}$$

is rewritten by using order statistics

$$\text{Minimize } \boxed{\sigma_{k:m}}\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}; \boldsymbol{\Theta}\right)$$

$$\text{subject to } \boxed{\gamma^{\text{cr}}_{k:m}}\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}; \boldsymbol{\Theta}\right) \le \gamma_{\text{U}};\quad V\left(\boldsymbol{x}_{\text{free}}(\boldsymbol{q}), \boldsymbol{y}_{\text{free}}(\boldsymbol{q}), \boldsymbol{A}\right) \le V_{\text{U}};$$

$$\underline{\boldsymbol{q}} \le \boldsymbol{q} \le \bar{\boldsymbol{q}};\quad \underline{\boldsymbol{A}} \le \boldsymbol{A} \le \bar{\boldsymbol{A}}$$

***k*th order statistics with sample size *m* and confidence level α_k**

Smaller order *k* → Larger robustness level

Penalization method for singularity phenomenon

Singularity in stress and stability

Small cross-sectional area → **Large element stress**

Small cross-sectional area → **Small linear buckling load**

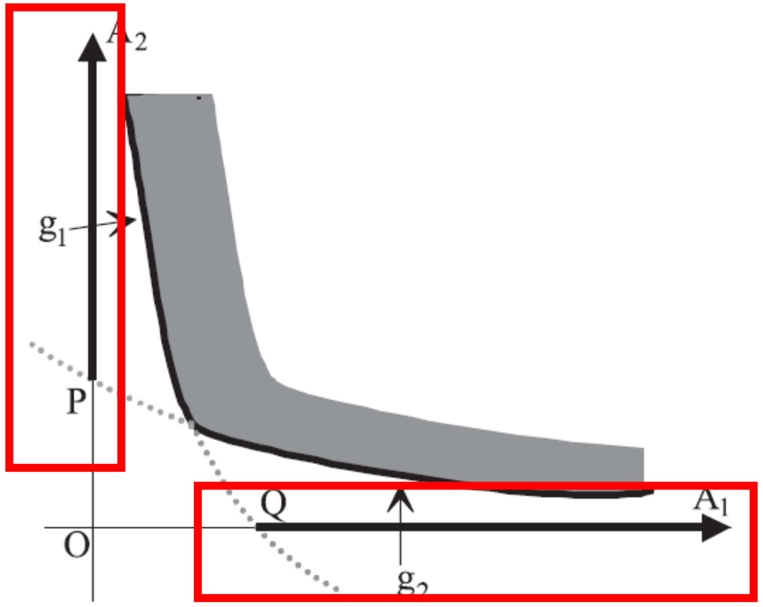


Fig. 2 Feasible domain under formulation (3)

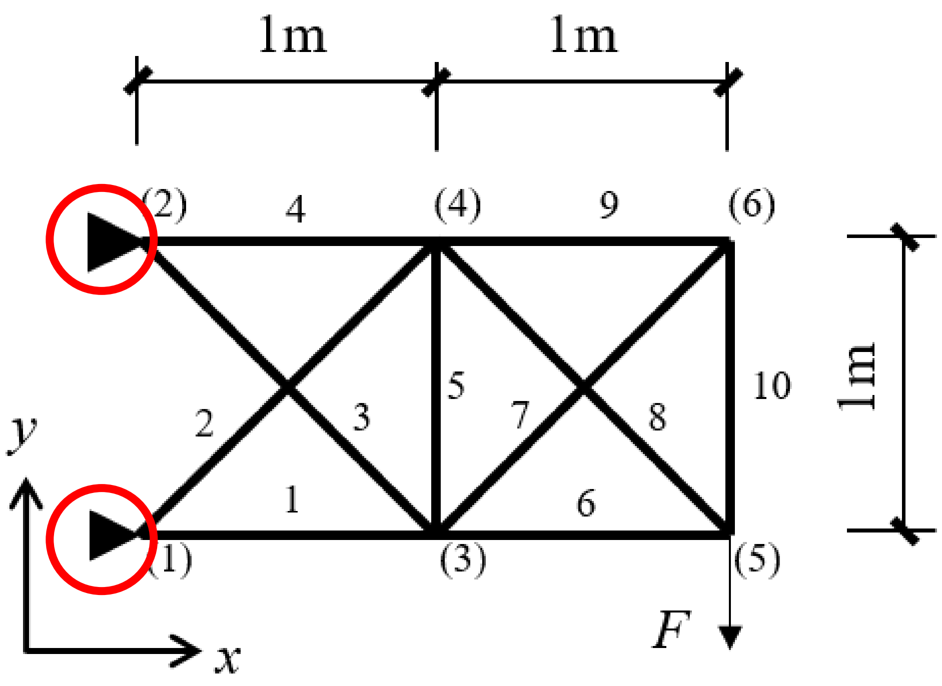
Penalization method for small *A*

$$\hat{\sigma}_{V,i} = \left(A_i / \bar{A}\right)^{\eta} \max_{j=1,2,\cdots p}\left(\sigma_{V,ij}\right) \qquad \hat{\boldsymbol{K}}_{G,i} = \left(A_i / \bar{A}\right)^{\rho} \times \boldsymbol{K}_{G,i}$$

Source: X Guo *et al*, Optimum design of truss topology under buckling constraints, 2005, Structural and Multidisciplinary Optimization

Numerical Example

- **Example:**



Pin support

Initial condition:

F = 200 kN; E = 2×10^{11} Pa; $V_U = 0.02\text{m}^3$

Sample size m = 150; k = 1;

Design variables:

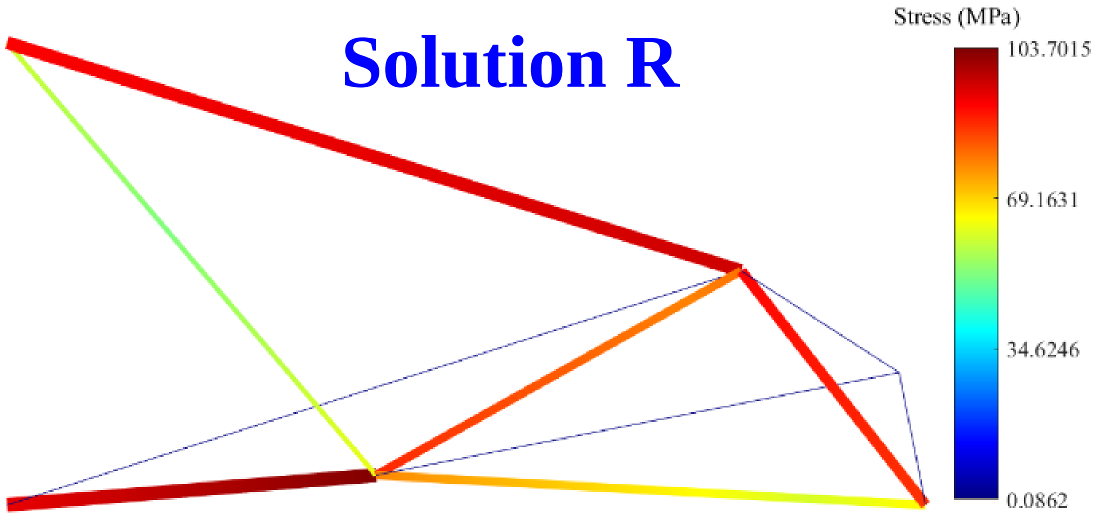
$\boldsymbol{A} = (A_1, A_2, \ldots, A_{10})$; $\boldsymbol{q} = (q_1, q_2, \ldots, q_{10})$

$\mathbf{1}\times10^{-7}\text{m}^2 \leq \boldsymbol{A} \leq \mathbf{0.05}\ \text{m}^2$; $\mathbf{-1000} \leq \boldsymbol{q} \leq \mathbf{1000}$

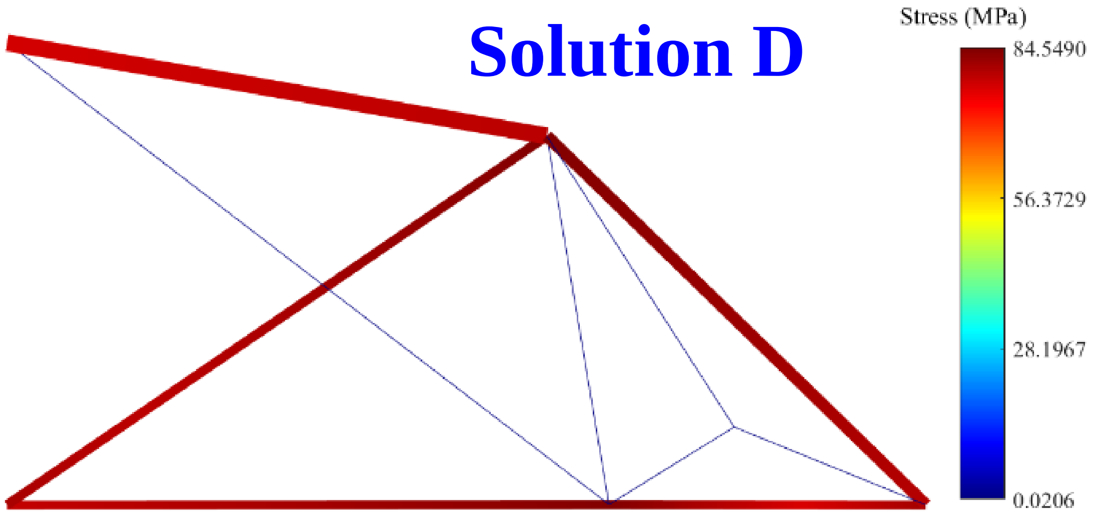
Solved by generalized reduced gradient (GRG) method

Numerical Example

Optimal results (at nominal condition)



Considering Uncertainty

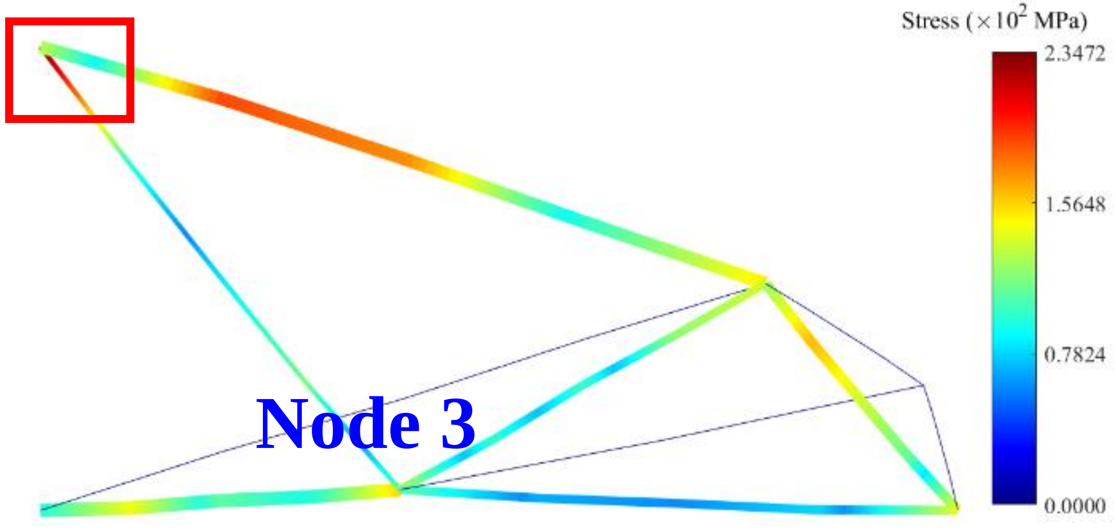


Without Uncertainty

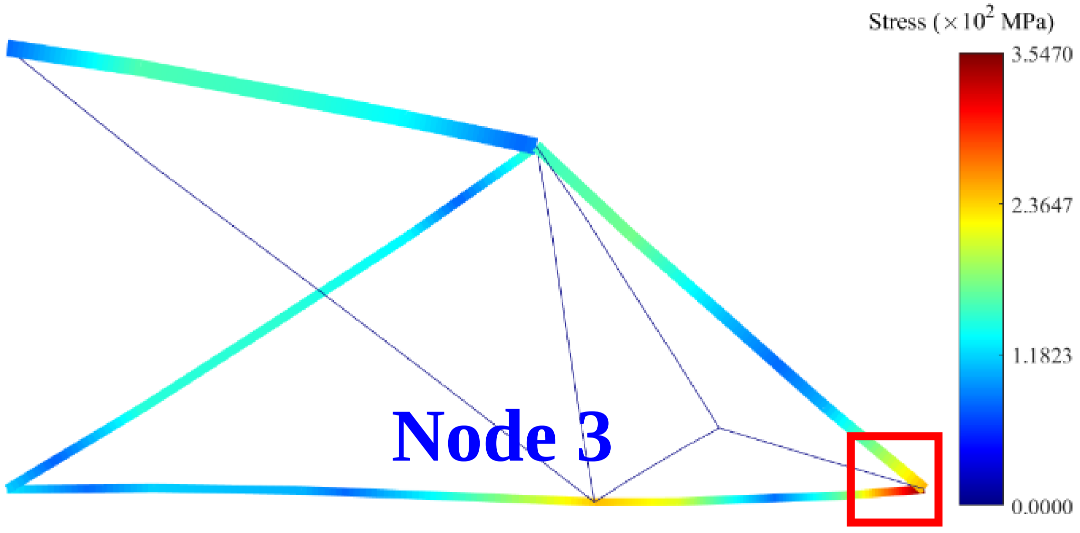
Solution	σ (MPa)	$\sigma_{1:150}^{\max}$ (MPa)	γ^{cr}	λ^{cr}	$\gamma_{1:150}^{cr}$	$\lambda_{1:150}^{cr}$	V (m^3)
R	103.7015	234.7208	0.0868	11.5117	0.0914	10.9328	0.02
D	84.5490	354.7003	0.2899	3.4490	0.3193	3.1314	0.02

Numerical Example

Optimal results (at worst case condition)



Solution R



Solution D

Solution	σ (MPa)	$\sigma_{1:150}^{max}$ (MPa)	γ^{cr}	λ^{cr}	$\gamma_{1:150}^{cr}$	$\lambda_{1:150}^{cr}$	V (m^3)
R	103.7015	234.7208	0.0868	11.5117	0.0914	10.9328	0.02
D	84.5490	354.7003	0.2899	3.4490	0.3193	3.1314	0.02

Conclusions and future work

- **The proposed method has the following conclusions:**

  - **The robustness of the objective and constraints are represented by the *k*th order statistics.**
  - **An auxiliary truss to which the FDM is applied is used to define the geometry of the frame.**
  - **The stress and geometrical stiffness matrix of a thin element is penalized with respect to the cross-sectional area.**
  - **Solutions with and without considering uncertainties may have different shapes and topologies.**

Conclusions and future work

- **Future work:**

- **Since the uncertainty is directly incorporated in the optimization problem, the analytical gradient is hard to derive. Maybe decoupling or approximation technique would be useful for improving its convergence and application on large-scale problem.**
- **The robustness is determined directly by the order statistics, and maybe summary statistics (such as statistical moments) would be helpful to give the similar information of uncertainty with smaller sample size.**

Thanks for your kind attention

京都大学
KYOTO UNIVERSITY

Reporter : Wei Shen